



**B.E. International Program**  
Faculty of Economics, Thammasat University



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**EE 320 Introductory Mathematical Economics**

**Semester 1/2017**

**Practice Problem 6**

**(Unconstrained Optimization Problems – Multiple-Independent Variables)**

**Question 1:** Suppose that there are two firms in the industry, and they are competing in quantities. The amount of the commodity sold by firm  $i$  is  $q_i$ ,  $i = 1, 2$ . The market demand function is given by  $P = 50 - 3q$ , where  $q = q_1 + q_2$ . The cost functions for each firm is given by  $TC_i = 25 + 5q_i$ ,  $i = 1, 2$ .

- Find the profit-maximizing quantity for each firm, and determine each firm's profit level.
- Suppose that both firms merge. Compute the new profit-maximizing quantity and the new profit of the merged firm. Do firms have incentive to merge, and why?

**Question 2:** Given the production function

$$Q = f(K, L) = 8K^{1/2}L^{1/4}$$

Suppose that the price per unit of  $Q$  is  $\$P$ , and the per unit input prices for  $K$  and  $L$  are  $\$r$  and  $\$w$ , respectively. ( $P$ ,  $w$ , and  $r$  are positive constants.)

- Solve for the values  $r$  and  $P$  that maximizes the profit. Verify that the second-order sufficient condition is met.
- Find the comparative statics derivatives  $\frac{\partial K^*}{\partial r}$  and  $\frac{\partial L^*}{\partial P}$ , evaluate the signs, and interpret their economic meanings.

**\*Question 3:** Each of two firms A and B produces its own brand of a commodity, such as mineral water, in amounts denoted by  $x$  and  $y$ , and these are sold at prices  $p$  and  $q$  unit, respectively. Each firm determines its own price and produces exactly as much as is demanded. The demands for the two brands are given by

$$x = 29 - 5p + 4q$$

$$y = 16 + 4p - 6q$$

Firm A has total costs  $5 + x$ , whereas firm B has total costs  $3 + 2y$ .

(Assume that the functions to be maximized have maxima, and at positive prices.)

- Initially, the two firms collude in order to maximize their combined profits, as one monopolist would. Find the prices  $(p, q)$ , the production levels  $(x, y)$ , and the profits of firms A and B.
- Then, an antitrust authority prohibits collusion, so each producer maximizes its own profit, taking the other's price as given.
  - If  $q$  is fixed, how will A choose  $p$ ? (Find  $p$  as a function  $p = p_A(q)$ .)
  - If  $p$  is fixed, how will B choose  $q$ ? (Find  $q$  as a function  $q = q_B(p)$ .)
- Under the assumptions in part b), what constant equilibrium prices are possible? What are the production levels and profits in this case?

**Question 4** A firm produces two different kinds A and B of a commodity.

The daily cost of producing  $Q_1$  units of A and  $Q_2$  units of B is:

$$C(Q_1, Q_2) = 0.1(Q_1^2 + Q_1Q_2 + Q_2^2) .$$

Suppose that the firm sells all its output at a price per unit  $P_1 = 120$  for A and  $P_2 = 90$  for B.

- Find the daily production levels that maximize profit.
- What prices ( $P_1$ ) per unit of A would imply that the optimal daily production level for A is 400 units?

### Question 5:

Let the total cost function depend on goods x, y and z ;

$$TC = 1,000 + 3x^2 + 2y^2 + 2z^2 - 2xy - 40z - 20x$$

Determine the level of x , y and z which minimize total cost and determine the minimum total cost.

### Question 6:

A monopolist produces two products, A, and B. The joint-cost function is  $C = 5000 + 5q_A + 3q_B$ , where c is the total cost of producing  $q_A$  units of A and  $q_B$  units of B. The demand functions for these products are given by  $p_A = 205 - 2q_A - q_B$  and  $p_B = 153 - q_A - q_B$ , where  $p_A$  and  $p_B$  are the prices of A and B, respectively. Consider the following problem.

- Determine the profit-maximizing level output for both products.
- How much should the monopolist set the price of the two products?

### Question 7:

A manufacturer produces products A and B for which the average costs of production are constant at 3 and 5 (dollars per unit), respectively. The quantities  $q_A$ ,  $q_B$  of A and B that can be sold each week are given by the joint-demand functions  $q_A = 10 - p_A + p_B$  and  $q_B = 12 + p_A - 3p_B$ ,  $p_A$  and  $p_B$  are the prices (in dollars per unit) of A and B, respectively. Determine the prices of A and B at which the manufacturer can maximize profit.

**Question 8:**

Consider a market with 2 firms. Each firm sells an identical product, facing the same market demand equation given by  $p = 10 - Q$ . For the first firm, denoted by firm 1, the cost function is given by  $C = c_1 Q_1$ . For the second firm, the cost function is given by  $C = c_1 Q_2^2$ . Consider the following problem.

- a. Determine the level of output that each firm will choose to produce under the Cournot equilibrium.
- b. State the requirement for  $c_1$  in order to ensure that both firms stay active in the equilibrium.
- c. Do they share the same market size under the equilibrium? Explain your result with some economic intuitions.
- d. Determine the level of output that each firm will produce under the collusion.

**Question 9:**

The profit function of a firm is  $\pi(x, y) = px + qy - \alpha x^2 - \beta y^2$ , where  $p$  and  $q$  are the prices per unit and  $\alpha x^2 + \beta y^2$  are the costs of producing  $x$  units of the first good and  $y$  units of the other. The constants are all positive.

- a. Find the values of  $x$  and  $y$  that maximize profits. Denote them by  $x^*$  and  $y^*$ . Verify that the second-order conditions are satisfied.
- b. Define  $\pi(p, q) = \pi(x^*, y^*)$  as the optimal profit function. The function generates the level of maximum profit that firm attain under different

combination of profit-maximizing output bundles. Verify that  $\partial\pi(p,q)/\partial p = x^*$  and  $\partial\pi(p,q)/\partial q = y^*$ . Give these results economic interpretations.

- c. Show that  $\pi(p,q)$  is convex in  $p$  and  $q$ . That is, you show that Hessian matrix of the optimal profit function is positive definiteness.

**Question 10:**

Define  $f(x,y)$  for all  $(x,y)$  by  $f(x, y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$

- Derive the Hessian matrix of  $f(x, y)$ .
- Show that  $f(x,y)$  is monotonically convex function. That's, the function is convex for the entire domain sets defining  $f(x,y)$ .
- Find the *global* extrema of  $f(x,y)$

**Question 11:**

Consider a function  $f(x, y) = x^2 - y^2 - xy - x^3$

- Find and classify the stationary points of  $f(x, y)$
- Find the domain set of  $f(x, y)$  where  $f(x, y)$  is concave, and find the largest value of  $f(x, y)$  in that domain set.