

FN211 Regression Analyses

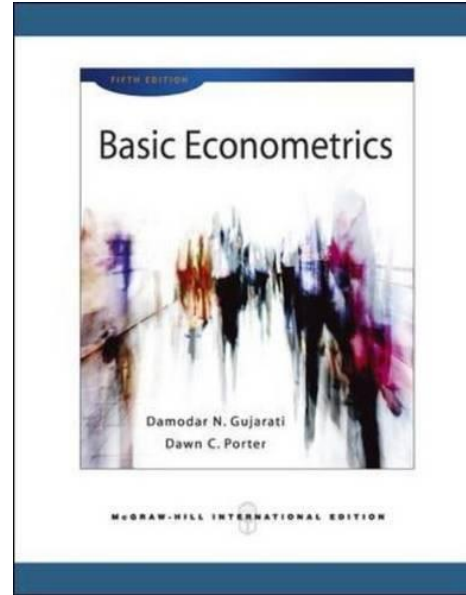
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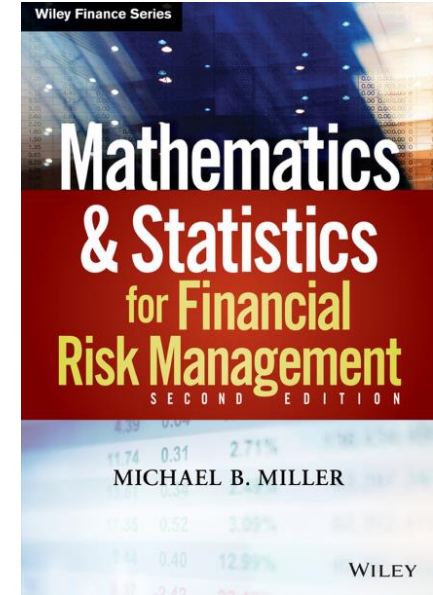
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Outline

- Simple Regression
- Multiple Regression
- Application



Reading: PART I



Reading: Chapter 10

Recall: Basic statistics

- Arithmetic Average:
$$E(r) = \sum_{s=1}^n p(s)r(s) = \frac{1}{n} \sum_{s=1}^n r(s)$$

- Variance:
$$\hat{\sigma}^2 = \frac{1}{n} \sum_{s=1}^n [r(s) - \bar{r}]^2$$

- Covariance:
$$\sigma_{XY} = \frac{\sum_{i=1}^n (X_i - \mu_X)(Y_i - \mu_Y)}{n}$$

Recall: Basic statistics

- Correlation:
$$\rho = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

What is regression?

- Correlation analysis VS. Regression analysis

“CAUSE VS EFFECT”

Independent variable VS Dependent variable

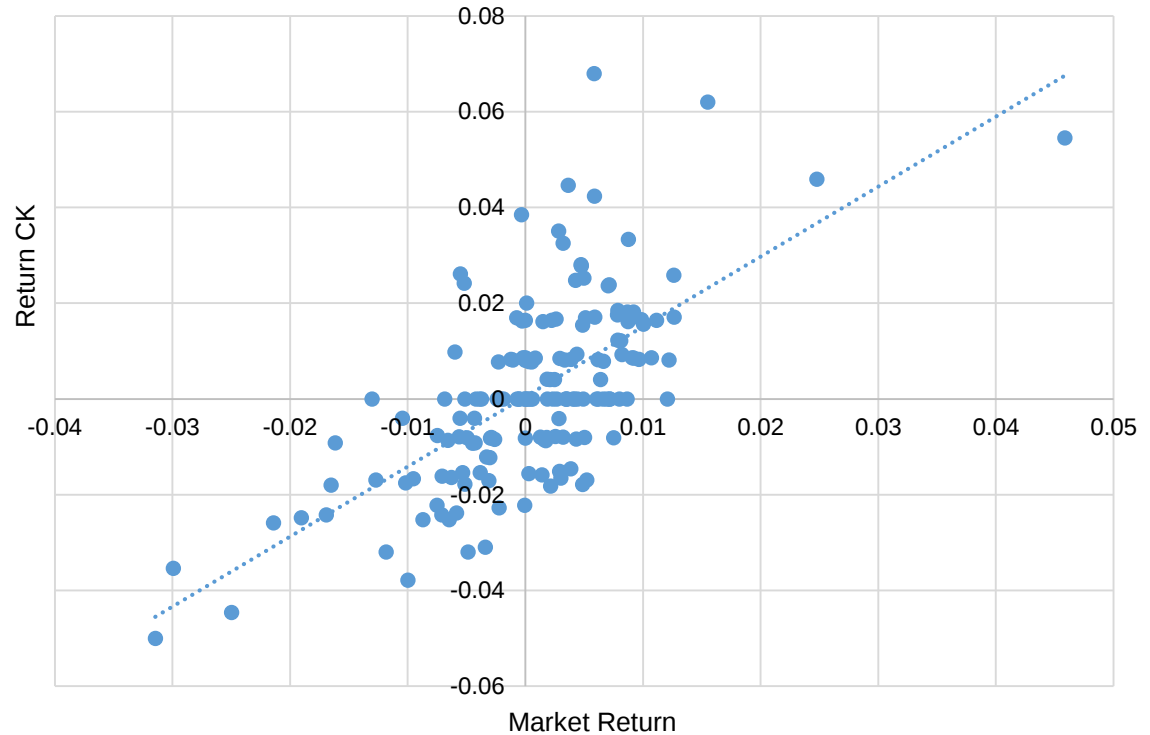
Example

1. Movement between returns of the 2 stocks during 1st quarter of 2019
2. The impact of SET index on ADVANCE stock return
3. Relation between Finance and Statistics scores of BE students

Intuition – Index Model

Estimation Model:

$$R_{Kbank} = \alpha + \beta R_m + \varepsilon_t$$



Classical Linear Regression Analysis

Simple Linear Regression:

$$y = \beta_0 + \beta_1 x_1 + \epsilon$$

Multiple Linear Regression:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \epsilon$$

(General form)

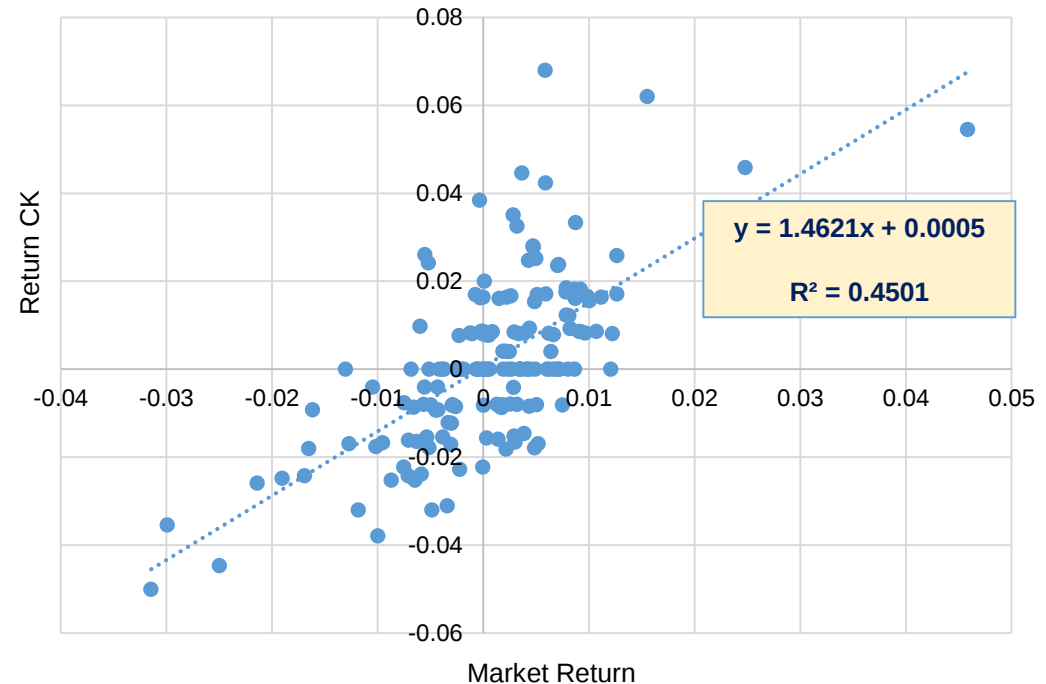
Simple Regression

The important parameter of estimation

- Coefficient
- Standard deviation
- R-square and Adjust R-square
(Test of Goodness of Testing)
- Z, t, F statistics
(Hypothesis Testing)

$$E(R_{Kbank} | R_m) = \alpha + \beta R_m$$

$$E(R_{Kbank} | R_m = 1) = \dots$$



Estimation & Interpretation – Ordinary Least Square (OLS)

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

Ordinary Least Square: Begin with **Population Regression Function (PRF)**

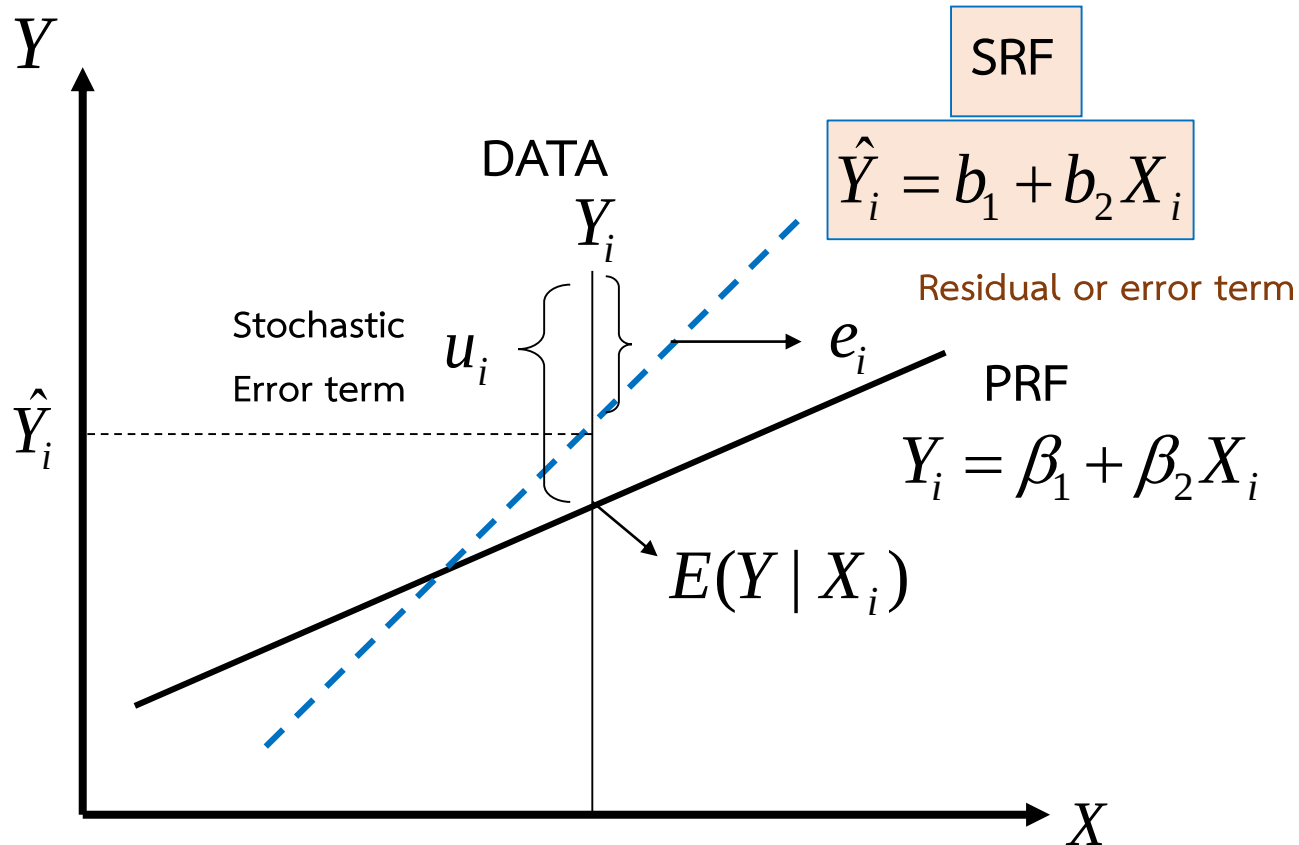
$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

However, PRF is not directly observable, we estimate it from **Sample Regression Function (SRF)**

$$Y_i = b_1 + b_2 X_i + e_i$$

$$Y_i = \hat{Y}_i + e_i$$

Estimation & Interpretation – Ordinary Least Square (OLS)



Estimation & Interpretation – Ordinary Least Square (OLS)

- This method creates the high probability that the estimators are the best estimation to the parameters
- It tries to minimize the residuals squares of estimation.

$$\min \sum_{i=1}^n e_i^2 = \min \sum_{i=1}^n (Y_i - \hat{Y}_i)^2$$

$$\min \sum_{i=1}^n e_i^2 = \min \sum_{i=1}^n (Y_i - b_1 - b_2 X_i)^2$$

Estimation & Interpretation – Ordinary Least Square (OLS)

OLS Estimators

$$\begin{aligned} b_2 &= \frac{n \sum X_i Y_i - \sum X_i \sum Y_i}{n \sum X_i^2 - (\sum X_i)^2} \\ &= \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} \\ &= \frac{\sum x_i y_i}{\sum x_i^2} \\ &= \frac{\sum x_i Y_i}{\sum X_i^2 - n\bar{X}^2} = \frac{\sum X_i y_i}{\sum X_i^2 - n\bar{X}^2} \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{\sum X_i^2 \sum Y_i - \sum X_i \sum X_i Y_i}{n \sum X_i^2 - (\sum X_i)^2} \\ &= \bar{Y} - b_2 \bar{X} \end{aligned}$$

Estimation & Interpretation – Ordinary Least Square (OLS)

Precision or Standard Errors of Least-Squares Estimates

Estimation of σ^2 :

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n [Y_i - (\hat{\beta}_1 + \hat{\beta}_2 X_i)]^2}{n - 2}$$

Number of parameter



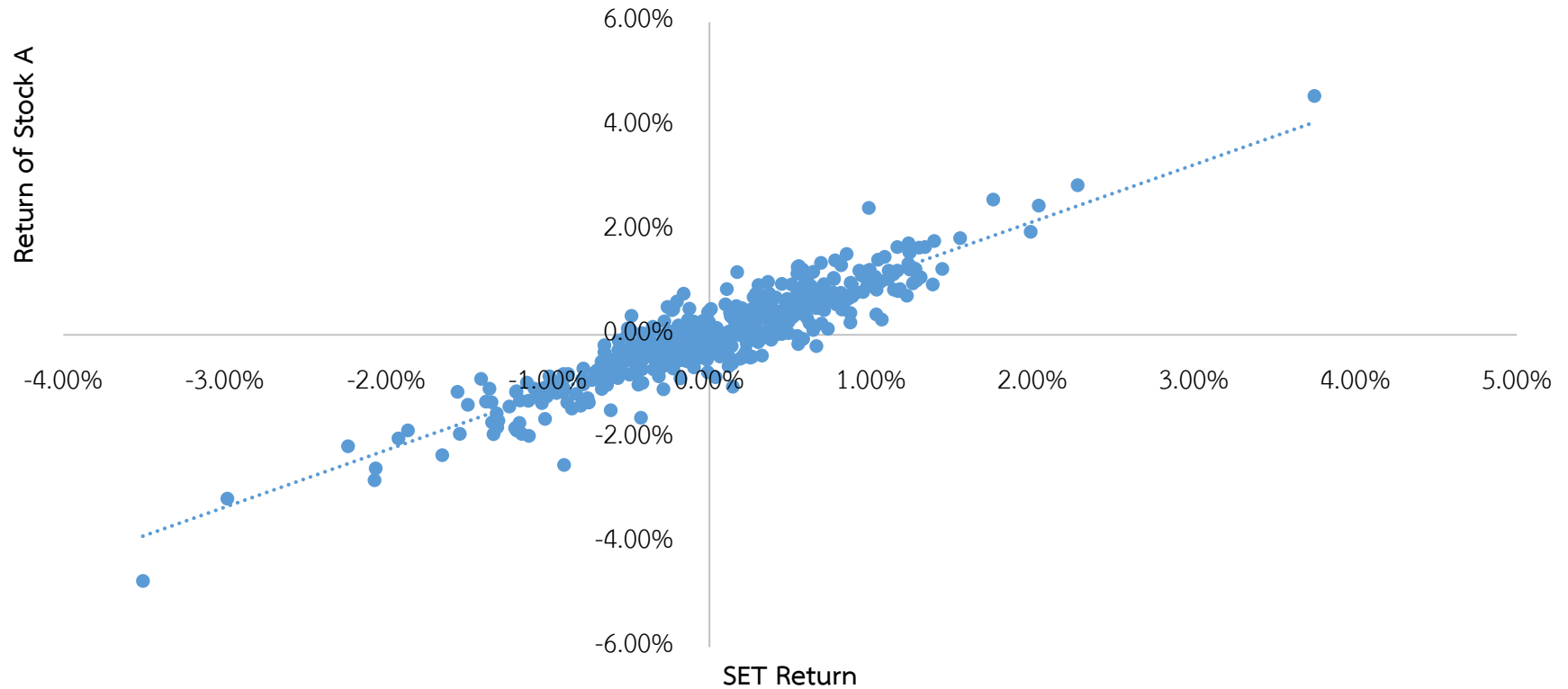
$$\text{var}(b_2) = \frac{\sigma^2}{\sum x_i^2}$$

$$\text{se}(b_2) = \frac{\sigma}{\sqrt{\sum x_i^2}}$$

$$\text{var}(b_1) = \frac{\sum X_i^2}{n \sum x_i^2} \sigma^2$$

$$\text{se}(b_1) = \sqrt{\frac{\sum X_i^2}{n \sum x_i^2} \sigma^2}$$

What is regression output?



Regression Output

SUMMARY OUTPUT						
<i>Regression Statistics</i>						
Multiple R	0.9159					
R Square	0.8389					
Adjusted R Square	0.8386					
Standard Error	0.0029					
Observations	492					
ANOVA						
	<i>df</i>	<i>SS</i>	<i>MS</i>	<i>F</i>	<i>Significance F</i>	
Regression	1	0.0221	0.0221	2552.4109	0.0000	
Residual	490	0.0042	0.0000			
Total	491	0.0263				
	<i>Coefficients</i>	<i>Standard Error</i>	<i>t Stat</i>	<i>P-value</i>	<i>Lower 95%</i>	<i>Upper 95%</i>
Intercept	0.0002	0.0001	1.71	0.0885	0.0000	0.0005
X Variable 1	0.7653	0.0151	50.52	0.0000	0.7356	0.7951

Hypothesis Testing and Interval Estimation

Test individual coefficient

test statistic to verify or falsify the null hypothesis

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu \neq \mu_0$$

Testing of the significance of regression coefficient: t - test

$$t = \frac{\overline{X} - \mu_0}{S_x / \sqrt{n}}$$

Hypothesis Testing and Interval Estimation

Confidence interval derivation

$$\Pr(-t_{\alpha/2} \leq t \leq t_{\alpha/2}) = 1 - \alpha$$

$100(1 - \alpha)\%$ confidence interval for β_1 and β_2

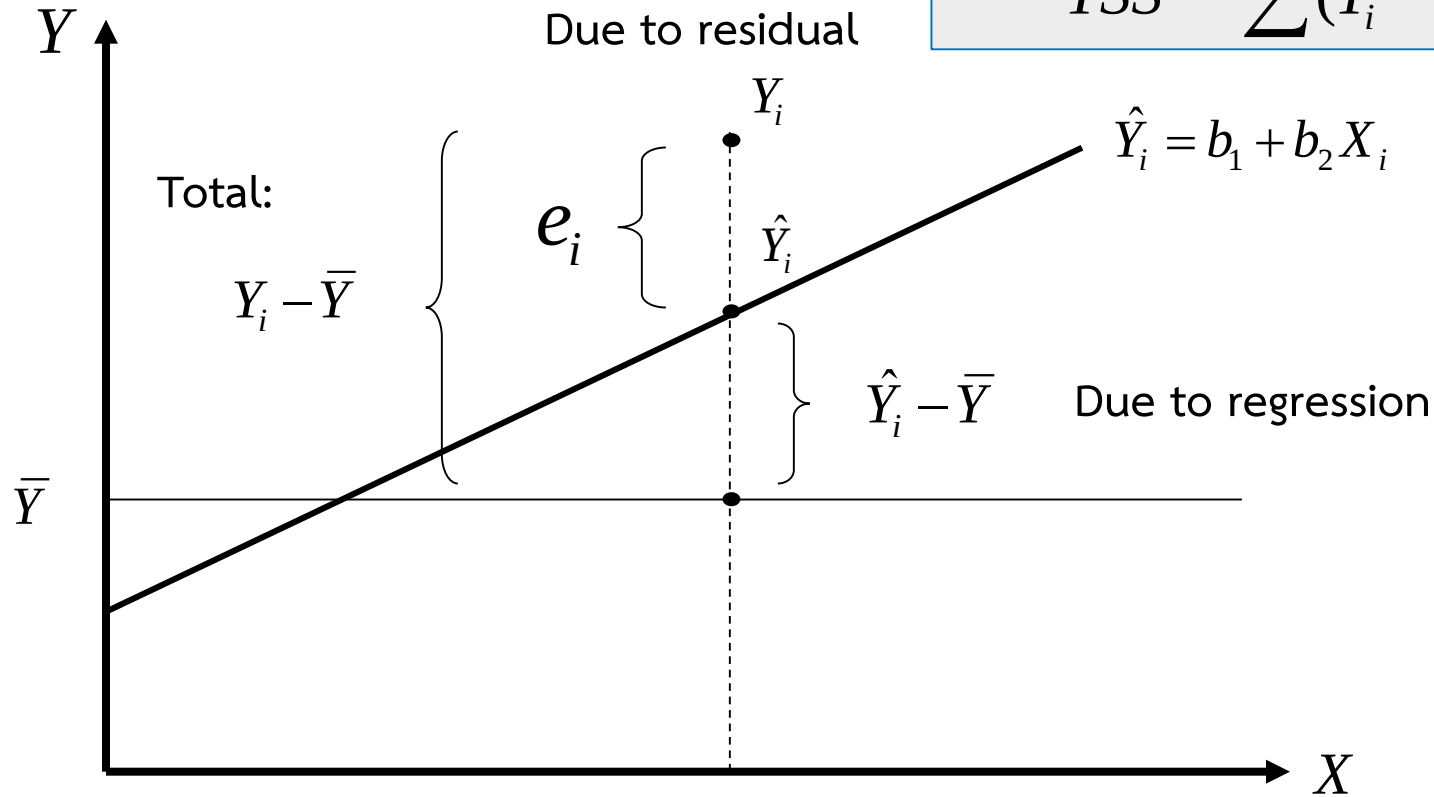
$$\hat{\beta}_1 \pm t_{\alpha/2} S_{\hat{\beta}_1}$$

$$\hat{\beta}_2 \pm t_{\alpha/2} S_{\hat{\beta}_2}$$

Estimation & Interpretation – Ordinary Least Square (OLS)

Coefficient of Determination

$$r^2 = \frac{ESS}{TSS} = \frac{\sum (\hat{Y}_i - \bar{Y})^2}{\sum (Y_i - \bar{Y})^2}$$



Estimation & Interpretation – Ordinary Least Square (OLS)

Coefficient of Determination: R^2 and Adjusted R^2

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum \hat{u}_i^2}{\sum y_i^2}$$
$$\bar{R}^2 = 1 - \frac{\sum \hat{u}_i^2 / (n - \boxed{k})}{\sum y_i^2 / (n - 1)}$$

number of parameters
in the model including
the intercept term

Adjust: Adjusted for the degree of freedom associated with the sum of square

$$\bar{R}^2 = 1 - (1 - R^2) \frac{(n - 1)}{(n - k)}$$

Estimation & Interpretation – Ordinary Least Square (OLS)

Analysis of Variance (ANOVA)

$$\begin{aligned}\sum y_i^2 &= \sum \hat{y}_i^2 + \sum e_i^2 \\ &= 2b_2^2 \sum X_i^2 + \sum e_i^2\end{aligned}$$

Source of Variation	SS	Df	MSS
Due to regression (ESS)	$\sum \hat{y}_i^2$	1	$b_2^2 \sum x_i^2$
Due to residual (RSS)	$\sum e_i^2$	$n - 2$	$\frac{\sum e_i^2}{n - 2} = \hat{\sigma}^2$
TSS	$\sum y_i^2$	$n - 1$	

$$F = \frac{\text{MSS of ESS}}{\text{MSS of RSS}}$$

Example

The relationship between nominal exchange rate and relative prices. From the annual observations from 1980 to 1994, the following regression results were obtained, where Y = exchange rate of the German mark to the U.S. dollar (GM/\$) and X = ratio of the U.S. consumer price index to the German consumer price index; that is, X represents the relative prices in the two countries:

$$\hat{Y}_t = 6.682 - 4.318X_t \quad r^2 = 0.528$$

$$se = (1.22)(1.333)$$

- a. Interpret this regression. How would you interpret r^2 ?
- b. Does the negative value of X_t make economic sense? What is the underlying economic theory?

Key Assumptions

1. Regressors are NOT correlated

2. Error variance is constant

$$\text{var}(u_i) = \sigma_i^2$$

3. Error terms are NOT correlated

$$E(\varepsilon_i, \varepsilon_j) \neq 0, \quad \text{cov}(\varepsilon_i, \varepsilon_j) \neq 0$$

4. Model is correct, i.e. liner vs. log form

5. Regressor and error term are correlated

$$E(X_i, \varepsilon_i) \neq 0, \quad \text{cov}(X_i, \varepsilon_i) \neq 0$$

Example

From a sample of 10 observations, the following results were obtained:

$$\sum Y_i = 1110 \quad \sum X_i = 1700 \quad \sum X_i Y_i = 205,500$$

$$\sum X_i^2 = 322,000 \quad \sum Y_i^2 = 132,100$$

with coefficient of correlation $r = 0.9758$. But on rechecking these calculations it was found that two pairs of observations were recorded:

<u>Y</u>	<u>X</u>		<u>Y</u>	<u>X</u>
90	120	instead of	80	110
140	220		150	210

What will be the effect of this error on r ? Obtain the correct r .

Multiple Regression

Estimation & Interpretation – OLS

The Generic Multiple Regression Model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \dots + \beta_k X_{ki} + \varepsilon_i \quad i = 1, \dots, n$$

Estimation of regression parameters:

The use of the matrix notation allows a view of how the data are housed in software programs.

$$Y_1 = \beta_1 + \beta_2 X_{21} + \beta_3 X_{31} + \dots + \beta_k X_{k1} + u_1$$

$$Y_2 = \beta_1 + \beta_2 X_{22} + \beta_3 X_{32} + \dots + \beta_k X_{k2} + u_2$$

.....

$$Y_n = \beta_1 + \beta_2 X_{2n} + \beta_3 X_{3n} + \dots + \beta_k X_{kn} + u_n$$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n \times 1} \quad X = \begin{bmatrix} 1 & X_{11} & X_{12} & \dots & X_{1k} \\ 1 & X_{21} & X_{22} & \dots & X_{2k} \\ \vdots & \vdots & \vdots & & \\ 1 & X_{n1} & X_{n2} & \dots & X_{nk} \end{bmatrix}_{n \times k} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}_{k \times 1} \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n \times 1}$$

Estimation & Interpretation – OLS

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{21} & X_{31} & \cdots & X_{k1} \\ 1 & X_{22} & X_{32} & \cdots & X_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2n} & X_{3n} & \cdots & X_{kn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$
$$\begin{matrix} \mathbf{y} & = & \mathbf{X} & \boldsymbol{\beta} & + & \mathbf{u} \\ n \times 1 & & n \times k & k \times 1 & & n \times 1 \end{matrix}$$

$$\begin{matrix} \mathbf{y} & = & \mathbf{X} & \boldsymbol{\beta} & + & \mathbf{u} \\ n \times 1 & & n \times k & k \times 1 & & n \times 1 \end{matrix}$$

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \cdots - \hat{\beta}_k X_{ki})^2$$

$$\hat{\mathbf{u}}' \hat{\mathbf{u}} = [\hat{u}_1 \quad \hat{u}_2 \quad \cdots \quad \hat{u}_n] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \hat{u}_1^2 + \hat{u}_2^2 + \cdots + \hat{u}_n^2 = \sum \hat{u}_i^2$$

Estimation & Interpretation – OLS

$$\begin{aligned}\text{From: } \hat{u}'\hat{u} &= (\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta}) \\ &= \mathbf{y}'\mathbf{y} - 2\hat{\beta}'\mathbf{X}'\mathbf{y} + \hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta}\end{aligned}$$

$$\text{FOC: } \frac{\partial(\hat{u}'\hat{u})}{\partial\hat{\beta}} = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\beta} = 0$$

$$(\mathbf{X}'\mathbf{X})\hat{\beta} = \mathbf{X}'\mathbf{y}$$

$$\text{Hence, } \hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}$$

Estimation & Interpretation – OLS

OLS Assumptions with matrix notation

Scalar notation	Matrix notation
1. $E(u_i) = 0$, for each i (3.2.1)	1. $E(\mathbf{u}) = \mathbf{0}$ where $\mathbf{u}$ and $\mathbf{0}$ are $n \times 1$ column vectors, $\mathbf{0}$ being a null vector
2. $E(u_i u_j) = 0$ $i \neq j$ (3.2.5) $= \sigma^2$ $i = j$ (3.2.2)	2. $E(\mathbf{u}\mathbf{u}') = \sigma^2 \mathbf{I}$ where $\mathbf{I}$ is an $n \times n$ identity matrix
3. $X_2, X_3, \dots, X_k$ are nonstochastic or fixed	3. The $n \times k$ matrix $\mathbf{X}$ is nonstochastic, that is, it consists of a set of fixed numbers
4. There is no exact linear relationship among the X variables, that is, no multicollinearity (7.1.7)	4. The rank of $\mathbf{X}$ is $p(\mathbf{X}) = k$, where k is the number of columns in $\mathbf{X}$ and k is less than the number of observations, n
5. For hypothesis testing, $u_i \sim N(0, \sigma^2)$ (4.2.4)	5. The $\mathbf{u}$ vector has a multivariate normal distribution, i.e., $\mathbf{u} \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$

Estimation & Interpretation – OLS

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

Substituting $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$ into the preceding expression gives

$$\begin{aligned}\hat{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + \mathbf{u}) \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \\ &= \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\end{aligned}$$

Therefore, $\hat{\beta} - \beta = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$

$$\begin{aligned}\text{var-cov}(\hat{\beta}) &= E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] \\ &= E\{[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}][(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}]'\} \\ &= E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\mathbf{u}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]\end{aligned}$$

$$\begin{aligned}\text{var-cov}(\hat{\beta}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\mathbf{u}\mathbf{u}')\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\end{aligned}$$

$$E(\mathbf{u}\mathbf{u}') = \sigma^2\mathbf{I}$$

Estimation & Interpretation – OLS

Coefficient of determination R-square:

$$R^2 = \frac{\hat{\beta}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2}{\mathbf{y}'\mathbf{y} - n\bar{Y}^2}$$

Testing the overall significance of regression:

$$F = \frac{(\hat{\beta}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2)/(k - 1)}{(\mathbf{y}'\mathbf{y} - \hat{\beta}'\mathbf{X}'\mathbf{y})/(n - k)}$$

ANOVA table for k -variable linear regression model:

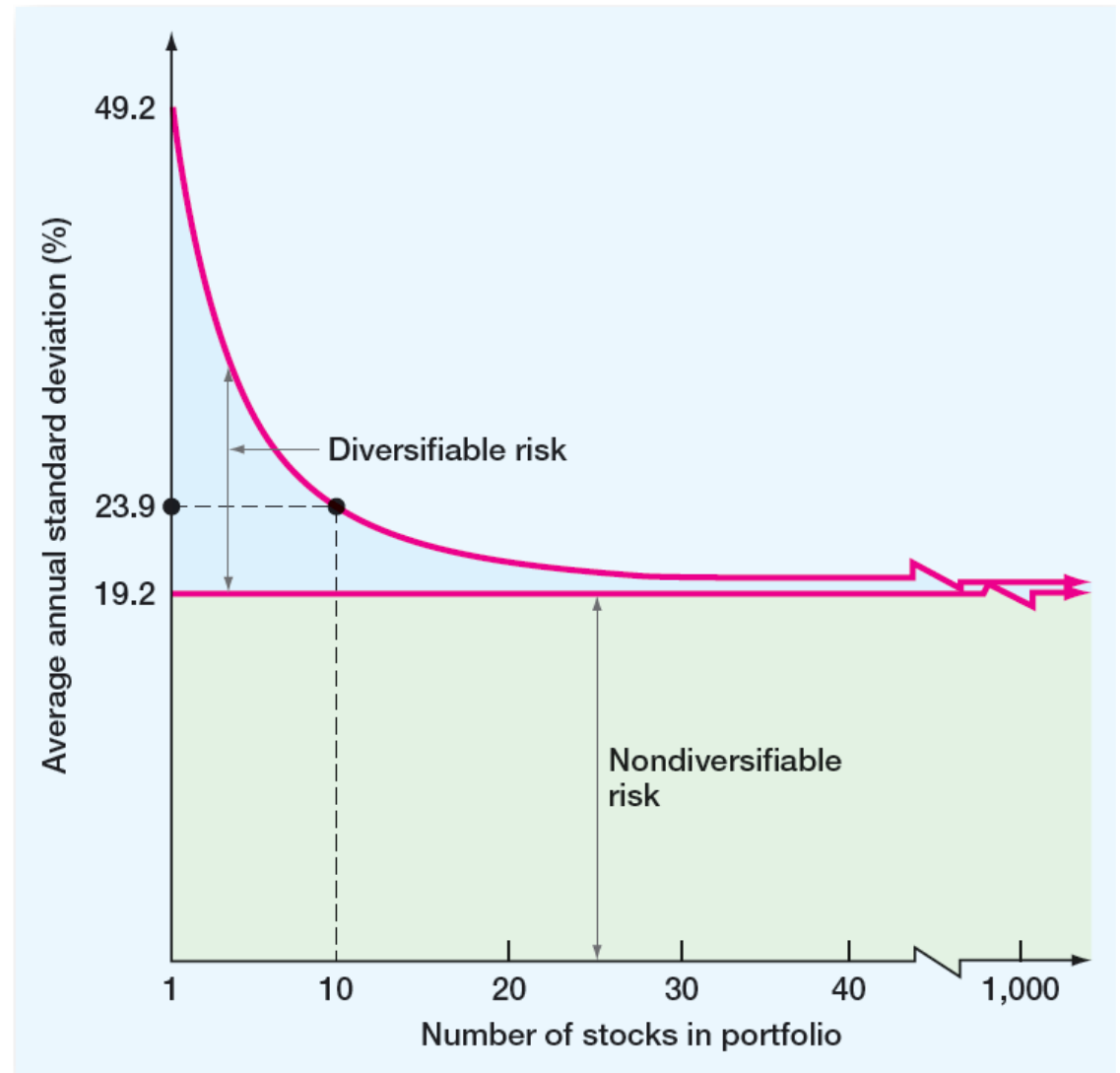
Source of variation	SS	df	MSS
Due to regression (that is, due to $X_2, X_3, \dots, X_k$)	$\hat{\beta}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2$	$k - 1$	$\frac{\hat{\beta}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2}{k - 1}$
Due to residuals	$\mathbf{y}'\mathbf{y} - \hat{\beta}'\mathbf{X}'\mathbf{y}$	$n - k$	$\frac{\mathbf{y}'\mathbf{y} - \hat{\beta}'\mathbf{X}'\mathbf{y}}{n - k}$
Total	$\mathbf{y}'\mathbf{y} - n\bar{Y}^2$	$n - 1$	

Application in Finance

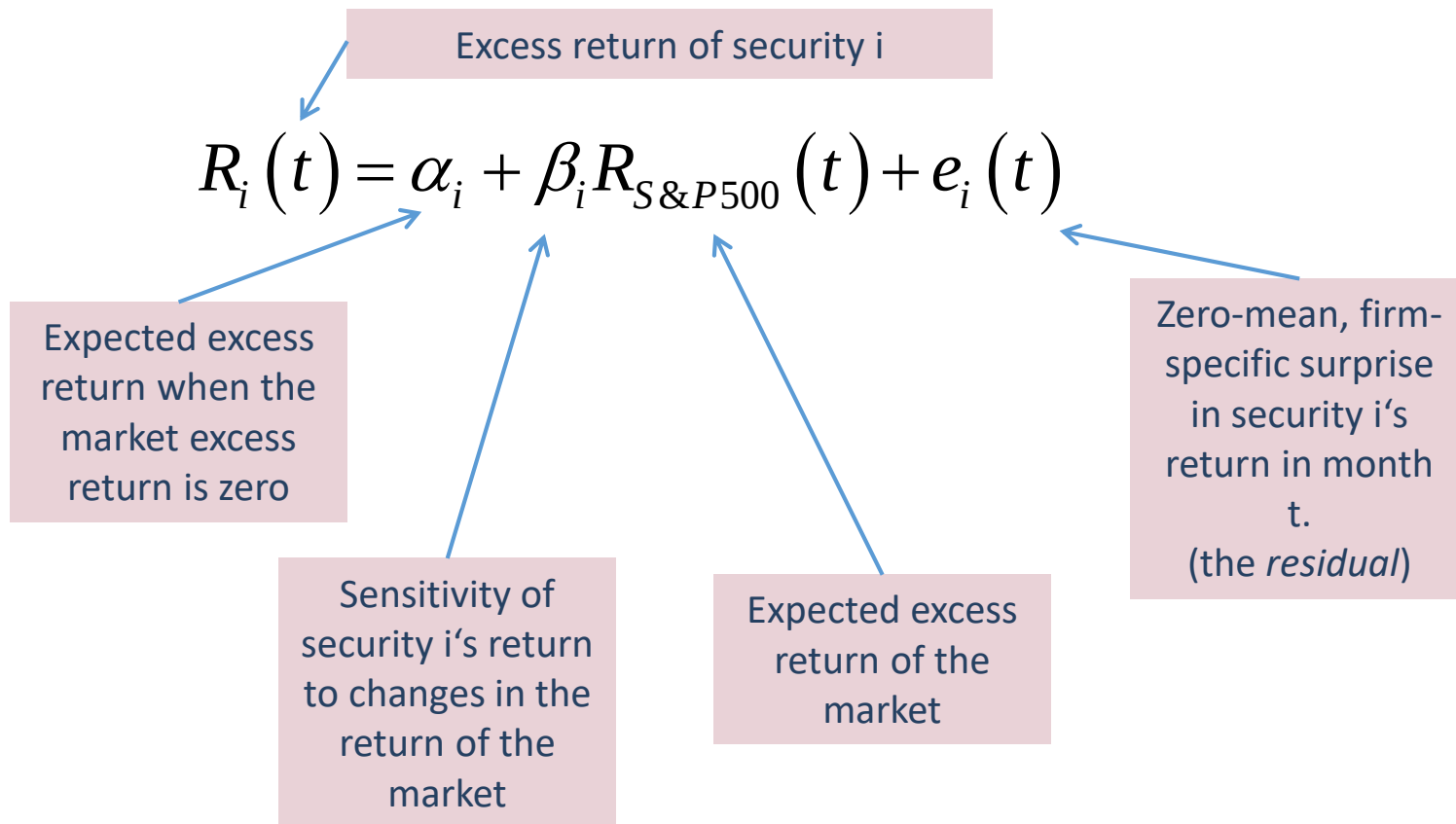
Recall: Total Risk

- Total risk = systematic risk + unsystematic risk
- The standard deviation of returns is a measure of total risk.
- For well-diversified portfolios, unsystematic risk is very small.
- Consequently, the total risk for a diversified portfolio is essentially equivalent to the systematic risk.

Recall: Portfolios



Index Model Regression Equation



Regression and Risk Management

- Simple Linear Regression:

$$y = \beta_0 + \beta_1 x_1 + \epsilon$$

OR

$$Y = \underbrace{\alpha + \beta X}_{\text{systematic}} + \underbrace{\epsilon}_{\text{idiosyncratic}}$$

- 1) A part that CAN be explained by the model
- 2) A part that CANNOT be explained by the model

Regression and Risk Management

- Variance = Systematic risk + Firm-specific risk:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i)$$

Why?

Question?