

EE320 Lecture Note

Chapter 7: Derivatives of More Than One Independent Variable Function

1 Partial Differentiation

1.1 First-order partial derivatives

Consider $y = f(x_1, x_2, \dots, x_n)$ where $x_1, x_2, \dots, x_n$ are independent of one another. Suppose only x_1 changes by Δx_1 , y will change accordingly by Δy (other x_s remain the same)

$$\text{So, } \frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1}$$

If $\Delta x \rightarrow 0$, consider a small change in x_1

$$f_1 \equiv \frac{\partial y}{\partial x} \equiv \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta y}{\Delta x} \text{ called partial differentiation.}$$

How much y will change when x_1 changes?

ex. 1 $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, find $f_1(1, 3)$ and $f_2(1, 3)$

$$f_1 = \frac{\partial f}{\partial x_1} = 6x_1 + x_2 \rightarrow f_1(1, 3) = 6(1) + (3) = 9$$

$$f_2 = \frac{\partial f}{\partial x_2} = x_1 + 8x_2 \rightarrow f_2(1, 3) = 6(1) + 8(3) = 30$$

ex. 2 $z = 9 - x^2 - y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = -2x \text{ holding } y \text{ constant : if } x \uparrow \text{ by 1 unit, } z \downarrow \text{ by 2 units.}$$

$$\frac{\partial z}{\partial y} = -2y \text{ holding } x \text{ constant : if } y \uparrow \text{ by 1 unit, } z \downarrow \text{ by 2 units.}$$

ex. 3 $y = (2x_1^2x_2 + x_1)(x_2 + 1)$ find $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$

$$\frac{\partial y}{\partial x_1} = (2x_1^2x_2 + x_1)\frac{\partial}{\partial x_1}(x_2 + 1) + (x_2 + 1)\frac{\partial}{\partial x_1}(2x_1^2x_2 + x_1)$$

$$= (2x_1^2x_2 + x_1)(0) + (x_2 + 1)(4x_1x_2 + 1)$$

$$= (x_2 + 1)(4x_1x_2 + 1)$$

$$\frac{\partial y}{\partial x_2} = (2x_1^2x_2 + x_1)\frac{\partial}{\partial x_2}(x_2 + 1) + (x_2 + 1)\frac{\partial}{\partial x_2}(2x_1^2x_2 + x_1)$$

$$= (2x_1^2x_2 + x_1)(1) + (x_2 + 1)(2x_1^2)$$

$$= 2x_1^2x_2 + x_1 + 2x_1^2x_2 + 2x_1^2$$

$$= 4x_1^2x_2 + x_1 + 2x_1^2$$

ex. 4 $Q = Q(K, L) = 6K^{\frac{1}{2}}L^{\frac{1}{2}}$

$$\frac{\partial Q}{\partial K} = MP_K = \sqrt[3]{\frac{L}{K}}$$

$$\frac{\partial Q}{\partial L} = MP_L = \sqrt[3]{\frac{K}{L}}$$

Gradient vector

The gradient vector of the function $f(x_1, x_2, \dots, x_n)$ is an n-vector of all partial derivatives:

$$\nabla f(x_1, x_2, \dots, x_n) = (f_1 \ f_2 \ \cdots \ f_n)$$

where $f_i \equiv \frac{\partial f}{\partial x_i}$

ex. $Q = aK^bL^{1-b}$ where $a > 0$ and $0 < b < 1$

$$\nabla Q(K, L) = \left(\frac{\partial Q}{\partial K} \quad \frac{\partial Q}{\partial L} \right) = [abK^{b-1}L^{1-b} \quad a(1-b)K^bL^{-b}]$$

(Application : Demand & Supply Model)

$$Q^d = a - bP \quad a, b > 0$$

$$Q^s = -c + dP \quad c, d > 0$$

$$\Rightarrow P^* = \frac{a+c}{b+d}$$

$$Q^* = \frac{ad-bc}{b+d}$$

Conduct comparative-static derivative:

$$[a] \frac{\partial P^*}{\partial a} = \frac{1}{b+d} > 0$$

$$\frac{\partial Q^*}{\partial a} = \frac{d}{b+d} > 0$$

$$[b] \frac{\partial P^*}{\partial b} = \frac{-(a+c)}{(b+d)^2} < 0$$

$$\begin{aligned} \frac{\partial Q^*}{\partial b} &= \frac{(b+d)(-c) - (ad-bc)(1)}{(b+d)^2} \\ &= \frac{-(a+c)d}{(b+d)^2} < 0 \end{aligned}$$

$$[c] \frac{\partial P^*}{\partial c} = \frac{1}{b+d} > 0$$

$$\frac{\partial Q^*}{\partial c} = \frac{-b}{b+d} < 0$$

$$[d] \frac{\partial P^*}{\partial d} = \frac{-(a+c)}{(b+d)^2} < 0$$

$$\frac{\partial Q^*}{\partial d} = \frac{(a+c)b}{(b+d)^2} > 0$$

(Application : Elasticity)

Consider $z = f(x, y)$, partial elasticity of z w.r.t. x and y are

$$\varepsilon_{zx} = \frac{\frac{\partial z}{\partial x}}{\frac{z}{x}} = \frac{\partial z}{\partial x} \cdot \frac{x}{z}$$

$$\varepsilon_{zy} = \frac{\frac{\partial z}{\partial y}}{\frac{z}{y}} = \frac{\partial z}{\partial y} \cdot \frac{y}{z}$$

when all variables are positive,

$$\varepsilon_{zx} = \frac{\partial \ln z}{\partial \ln x}$$

$$\varepsilon_{zy} = \frac{\partial \ln z}{\partial \ln y}$$

Note $\frac{\partial \ln z}{\partial \ln x} = \frac{\frac{\partial z}{\partial x}}{\frac{z}{x}} = \frac{\partial z}{\partial x} \cdot \frac{x}{z} \cdot \frac{d \ln x}{dx} = \frac{1}{x}$

For n variables, $f(x_1, x_2, \dots, x_n)$:

$$\varepsilon_{zi} = \frac{\partial \ln z}{\partial \ln x_i} = \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial \ln x_i} \cdot \frac{x_i}{f(x_1, \dots, x_n)}$$

Elasticities of demand

Given demand function $Q_1 = \alpha - \beta P_1 + \gamma P_2 + \mu Y$ or $Q_1 = f(P_1, P_2, Y)$

(Note: P_2 = price of substitute (goods 2) and Y = income)

Own price elasticities of demand $\varepsilon_d = \frac{\partial Q_1}{\partial P_1} \cdot \frac{P_1}{Q_1} = (-\beta) \cdot \frac{P_1}{Q_1}$

Income elasticities of demand $\varepsilon_Y = \frac{\partial Q_1}{\partial Y} \cdot \frac{Y}{Q_1} = \mu \cdot \frac{Y}{Q_1}$

Cross-price elasticities of demand $\varepsilon_c = \frac{\partial Q_1}{\partial P_2} \cdot \frac{P_2}{Q_1} = (\gamma) \cdot \frac{P_2}{Q_1}$

ex. $Q_1 = 100 - P_1 + 0.75P_2 - 0.25P_3 + 0.0075Y$

At $Y = 10,000$, $P_1 = 10$, $P_2 = 20$, $P_3 = 40$ and $Q_1 = 100$

$$\varepsilon_d = \frac{\partial Q_1}{\partial P_1} \cdot \frac{P_1}{Q_1} = (-1)\left(\frac{10}{100}\right) = 0.1$$

$$\varepsilon_Y = \frac{\partial Q_1}{\partial Y} \cdot \frac{Y}{Q_1} = (0.0075)\left(\frac{10,000}{100}\right) = 0.75$$

$$\varepsilon_{12} = \frac{\partial Q_1}{\partial P_2} \cdot \frac{P_2}{Q_1} = 0.75\left(\frac{20}{100}\right) = 0.15 \quad \text{substitute}$$

$$\varepsilon_{13} = \frac{\partial Q_1}{\partial P_3} \cdot \frac{P_3}{Q_1} = -0.25\left(\frac{40}{100}\right) = -0.1 \quad \text{complement.}$$

Output Elasticities

Given linearly homogeneous Cobb-Douglas production function

$$Q = F(K, L) = AK^\alpha L^\beta; A, \alpha, \beta \neq 0$$

$$\text{Output elasticity of capital} = \varepsilon_{QK} = \frac{\partial Q}{\partial K} \cdot \frac{K}{Q} = \alpha$$

$$\text{Output elasticity of labor} = \varepsilon_{QL} = \frac{\partial Q}{\partial L} \cdot \frac{L}{Q} = \beta$$

ex. $Q = 6K^{\frac{1}{2}}L^{\frac{1}{2}}$

$$\begin{aligned} \varepsilon_{QK} &= \frac{\partial}{\partial K} 6K^{\frac{1}{2}}L^{\frac{1}{2}} \cdot \frac{K}{Q} = 6L^{\frac{1}{2}} \cdot \frac{1}{2}K^{-\frac{1}{2}} \cdot \frac{K}{Q} = 3L^{\frac{1}{2}}K^{-\frac{1}{2}} \cdot \frac{K}{6K^{\frac{1}{2}}L^{\frac{1}{2}}} = \frac{1}{2} \\ \varepsilon_{QL} &= \frac{\partial}{\partial L} 6K^{\frac{1}{2}}L^{\frac{1}{2}} \cdot \frac{L}{Q} = 6K^{\frac{1}{2}} \cdot \frac{1}{2}L^{-\frac{1}{2}} \cdot \frac{L}{Q} = 3K^{\frac{1}{2}}L^{-\frac{1}{2}} \cdot \frac{L}{6K^{\frac{1}{2}}L^{\frac{1}{2}}} = \frac{1}{2} \end{aligned}$$

(Application : Production Function)

Given a production function $Q = 36KL - 2K^2 - 3L^2$

$$MP_K = \frac{\partial Q}{\partial K} = 36L - 4K$$

$$MP_L = \frac{\partial Q}{\partial L} = 36K - 4L$$

If MR at $L = 3$ and $K = 2$ is \$ 5 ; $MRP_{L=3} = ?$, $MRP_{K=2} = ?$

$$MRP_{L=3} = MR \cdot MP_L = (5) \cdot [36(2) - 6(3)] = 270$$

$$MRP_{K=2} = MR \cdot MP_K = (5) \cdot [36(3) - 4(2)] = 500$$

$\leftrightarrow$ If $K \uparrow$ by 1 unit, the additional revenue is \$ 500.

If $L \uparrow$ by 1 unit, the additional revenue is \$ 270.

(Application : Multipliers in Macroeconomic Model)

Consider a national-income model

$$Y = C + I_0 + G$$

$$C = C_0 + c(Y - T) \quad C_0 > 0, 0 < c < 1$$

$$T = T_0 + tY \quad T_0 > 0, 0 < t < 1$$

$$Y^* = \frac{C_0 - cT_0 + I_0 + G_0}{1 - c - ct}$$

Conduct comparative static derivative ;

$$\frac{\partial Y^*}{\partial G_0} = \frac{1}{1-c+ct} \quad \text{government-expenditure multiplier.}$$

$$\frac{\partial Y^*}{\partial T_0} = \frac{-c}{1-c+ct} \quad \text{nonincome-tax multiplier.}$$

$$\begin{aligned} \frac{\partial Y^*}{\partial t} &= \frac{(1-c+ct)(0)-(C_0-cT_0+I_0+G_0)(c)}{(1-c+ct)^2} \\ &= \frac{-(C_0-cT_0+I_0+G_0)(c)}{(1-c+ct)(1-c+ct)} \\ &= \frac{-c}{(1-c+ct)} \cdot Y^* < 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial Y^*}{\partial c} &= \frac{(1-c+ct)(-T_0)-(C_0-cT_0+I_0+G_0)(-1+t)}{(1-c+ct)^2} \\ &= \frac{-T_0+cT_0-ctT_0+C_0-cT_0+I_0+G_0-tC_0+ctT_0-tI_0-tG_0}{(1-c+ct)^2} \\ &= \frac{C_0-tC_0+I_0+G_0-T_0tI_0-tG_0}{(1-c+ct)^2} \\ &= \frac{(1-t)(C_0+I_0+G_0-T_0)}{(1-c+ct)^2} > 0 \end{aligned}$$

$$\frac{\partial Y^*}{\partial I_0} = \frac{\partial Y^*}{\partial G_0} = \frac{\partial Y^*}{\partial C_0} = \frac{1}{1-c+ct} \quad \Delta \text{autonomous expenditure}$$

1.2 Second-order partial derivatives

Consider $z = f(x, y)$ which gives rise to:

$$f_x \equiv \frac{\partial z}{\partial x} \quad \text{and} \quad f_y \equiv \frac{\partial z}{\partial y}$$

“rate of change of z after x (y) changes”

We can find rate of change of f_x with respect to z , given fixed y by a second-order partial derivative:

- with respect to x

$$f_{xx} \equiv \frac{\partial}{\partial x} f_x \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right)$$

- with respect to y

$$f_{yy} \equiv \frac{\partial}{\partial y} f_y \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right)$$

We can also find cross partial derivatives

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right)$$

$$f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$$

Young's Theorem : let $y = f(x_1, \dots, x_n)$ is twice differentiable, then

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_j \partial x_i}; i \neq j$$

$$\therefore f_{xy} = f_{yx}$$

$$\text{ex. } Q = aK^b L^{1-b} = bK^{\frac{1}{4}} L^{\frac{3}{4}}$$

$$MP_K = \frac{\partial Q}{\partial K} = 6 \cdot \frac{1}{4} K^{-\frac{3}{4}} L^{\frac{3}{4}} = \frac{3}{2} \left(\frac{L}{K} \right)^{\frac{3}{4}} = Q_K$$

$$MP_L = \frac{\partial Q}{\partial L} = 6 \cdot \frac{3}{4} K^{\frac{1}{4}} L^{-\frac{1}{4}} = \frac{9}{2} \left(\frac{K}{L} \right)^{\frac{1}{4}} = Q_L$$

$$\frac{\partial MP_K}{\partial K} = \frac{\partial}{\partial K} \left(\frac{\partial Q}{\partial K} \right) = \frac{\partial}{\partial K} \left(\frac{3}{2} \left(\frac{L}{K} \right)^{\frac{3}{4}} \right) = \frac{-9}{8} L^{\frac{3}{4}} K^{-\frac{7}{4}} = Q_{KK} \text{ (diminishing MP of capital)}$$

$$\frac{\partial MP_L}{\partial L} = \frac{\partial}{\partial L} \left(\frac{\partial Q}{\partial L} \right) = \frac{\partial}{\partial L} \left(\frac{9}{2} \left(\frac{K}{L} \right)^{\frac{1}{4}} \right) = \frac{-9}{8} L^{-\frac{5}{4}} K^{\frac{1}{4}} = Q_{LL} \text{ (diminishing MP of labor)}$$

$$Q_{KL} = \frac{\partial}{\partial K} \left(\frac{\partial Q}{\partial L} \right) = \frac{\partial}{\partial K} \left(\frac{9}{2} \left(\frac{K}{L} \right)^{\frac{1}{4}} \right) = \frac{9}{8} L^{-\frac{1}{4}} K^{-\frac{3}{4}}$$

$$Q_{LK} = \frac{\partial}{\partial L} \left(\frac{\partial Q}{\partial K} \right) = \frac{\partial}{\partial L} \left(\frac{3}{2} \left(\frac{L}{K} \right)^{\frac{3}{4}} \right) = \frac{9}{8} L^{-\frac{1}{4}} K^{-\frac{3}{4}}$$

$$Q_{KL} = Q_{LK}$$

Hessian Matrix: The second-order partial derivatives can be written in a matrix form

$$\nabla^2 f(x) = H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

$$\text{for } f(x_1, x_2) \quad \nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

ex. $Q = aK^{\frac{1}{4}}L^{\frac{3}{4}} = Q(K, L)$

$$H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LK} & Q_{LL} \end{bmatrix} = \begin{bmatrix} \frac{-9}{8}L^{\frac{3}{4}}K^{\frac{-7}{4}} & \frac{9}{8}L^{\frac{-1}{4}}K^{\frac{-3}{4}} \\ \frac{9}{8}L^{\frac{-1}{4}}K^{\frac{-3}{4}} & \frac{-9}{8}L^{\frac{-5}{4}}K^{\frac{1}{4}} \end{bmatrix} \text{ symmetric matrix}$$

ex. $z = x^2 \cdot e^{-y}$ find f_x, f_y , and Hessian matrix

$$1^{st} \text{ order: } \frac{\partial z}{\partial x} = 2xe^{-y} \quad \frac{\partial z}{\partial y} = -x^2e^{-y}$$

$$2^{nd} \text{ order : } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (-x^2e^{-y}) = -2xe^{-y}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (2xe^{-y}) = -2xe^{-y}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (2xe^{-y}) = -2xe^{-y}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} (2xe^{-y}) = 2e^{-y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} (-x^2e^{-y}) = x^2e^{-y}$$

$$H = \begin{bmatrix} 2e^{-y} & -2xe^{-y} \\ -2xe^{-y} & x^2e^{-y} \end{bmatrix}$$

(Application: Agricultural Production Function)

Consider an agricultural production function.

$$Q = F(K, L, T) = AK^\alpha L^\beta T^\gamma; A > 0, \alpha > 0, \beta > 0, \gamma > 0.$$

where K = capital

L = labor

T = land

1st - order

$$MP_K = \alpha(AK^{\alpha-1}L^\beta T^\gamma)$$

$$MP_L = \beta(AK^\alpha L^{\beta-1}T^\gamma)$$

$$MP_T = \gamma(AK^\alpha L^\beta T^{\gamma-1})$$

2nd - order

$$\frac{\partial MP_K}{\partial K} = \alpha(\alpha - 1)AK^{\alpha-2}L^\beta T^\gamma$$

$$\frac{\partial MP_K}{\partial L} = \alpha(\beta)AK^{\alpha-1}L^{\beta-1}T^\gamma$$

$$\frac{\partial MP_K}{\partial T} = \alpha(\gamma)AK^{\alpha-1}L^\beta T^{\gamma-1}$$

$$\frac{\partial MP_L}{\partial K} = \alpha(\beta)AK^{\alpha-1}L^{\beta-1}T^\gamma$$

$$\frac{\partial MP_L}{\partial L} = \beta(\beta - 1)AK^\alpha L^{\beta-2}T^\gamma$$

$$\frac{\partial MP_L}{\partial T} = \beta(\alpha)AK^\alpha L^{\beta-1}T^{\gamma-1}$$

$$\frac{\partial MP_T}{\partial K} = \alpha(\gamma)AK^{\alpha-1}L^\beta T^{\gamma-1}$$

$$\frac{\partial MP_T}{\partial L} = \beta(\gamma)AK^\alpha L^{\beta-1}T^{\gamma-1}$$

$$\frac{\partial MP_T}{\partial T} = \gamma(\gamma - 1)AK^\alpha L^\beta T^{\gamma-2}$$

$$H = \begin{bmatrix} \alpha(\alpha - 1)AK^{\alpha-2}L^\beta T^\gamma & \alpha(\beta)AK^{\alpha-1}L^{\beta-1}T^\gamma & \alpha(\gamma)AK^{\alpha-1}L^\beta T^{\gamma-1} \\ \alpha(\beta)AK^{\alpha-1}L^{\beta-1}T^\gamma & \beta(\beta - 1)AK^\alpha L^{\beta-2}T^\gamma & \beta(\alpha)AK^\alpha L^{\beta-1}T^{\gamma-1} \\ \alpha(\gamma)AK^{\alpha-1}L^\beta T^{\gamma-1} & \beta(\gamma)AK^\alpha L^{\beta-1}T^{\gamma-1} & \gamma(\gamma - 1)AK^\alpha L^\beta T^{\gamma-2} \end{bmatrix}$$

Recall definition of derivative; $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$

Recall definition of derivative; $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$

$$\Rightarrow dy = f'(x)dx$$

where dy = differential of y tells approximate change in y

dx = differential of x tells approximate change in x

So, $\frac{dy}{dx}$: derivative	$\leftrightarrow$	differentiation w.r.t. x
dy : differential of y	$\leftrightarrow$	differentiation
product		process

Total Differentials

Total Differentials is sum of the approximated changes from all parameters

For $y = f(x_1, x_2)$

$$dy = f_1 dx_1 + f_2 dx_2 \quad \text{where } f_1 = \frac{\partial f}{\partial x_1}$$

$$f_2 = \frac{\partial f}{\partial x_2}$$

$$\begin{array}{lcl} \text{ex.} & y &= 2x_1 + 3x_2 \\ & dy &= 2dx_1 + 3dx_2 \approx \Delta y = 2\Delta x_1 + 3\Delta x_2 \end{array}$$

✓ For utility function with n variables : $U = U(x_1, ..., x_n)$

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \dots + \frac{\partial U}{\partial x_n} dx_n$$

$$dU = U_1 dx_1 + \dots + U_n dx_n = \sum_{i=1}^n U_i dx_i$$

Consider $n = 2$, $U(x_1, x_2)$

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2$$

Holding U constant $\Rightarrow dU = 0$

$$0 = MU_1 dX_1 + MU_2 dx_2$$

$$-MU_2 dx_2 = MU_1 dX_1 \Rightarrow \frac{dx_2}{dx_1} = -\frac{MU_1}{MU_2} = \text{MRS} = \text{Slope of IC}$$

ex. $U = x^\alpha y^\beta$, find MRS

$$dU = (\alpha x^{\alpha-1} y^\beta) dx + (\beta x^\alpha y^{\beta-1}) dy$$

$$0 = \alpha x^{\alpha-1} y^\beta dx + \beta x^\alpha y^{\beta-1} dy$$

$$\frac{dy}{dx} = -\frac{(\alpha x^{\alpha-1} y^\beta)}{\beta x^\alpha y^{\beta-1}} = -\frac{\alpha y}{\beta x}$$

✓ For production function : $Q = F(K, L)$

$$dQ = \frac{\partial F}{\partial K} \cdot dK + \frac{\partial F}{\partial L} \cdot dL$$

$$dQ = \underbrace{F_K}_{MP_K} dK + \underbrace{F_L}_{MP_L} dL$$

Holding products constant $\Rightarrow dQ = 0$

$$0 = MP_K dK + MP_L dL$$

$$\frac{dK}{dL} = -\frac{MP_L}{MP_K} = \text{MRTS} = \text{Slope of isoquant}$$

ex. $Q = 60K^{0.25}L^{0.75}$, find MRTS

$$\frac{dK}{dL} = -\frac{MP_L}{MP_K} = \frac{-4.5K^{0.25}L^{-0.25}}{1.5K^{-0.75}L^{0.75}} = \frac{-3K}{L}$$

3 Total Derivatives

Consider the function that independent variables are related one another.

ex. $y = f(x, w)$ where $x = g(w)$, or $y = f(g(w), w)$. Find $\frac{dy}{dw}$

$$1. \text{ By chain rule: } \frac{dy}{dw} = \frac{\partial f}{\partial x} \frac{dx}{dw} + \frac{\partial f}{\partial w} = \overbrace{f_x \frac{dx}{dw}}^{\text{indirect}} + \overbrace{f_w}^{\text{direct}}$$

$$2. \text{ Total differential: } dy = \frac{\partial y}{\partial x} \cdot dx + \frac{\partial y}{\partial w} \cdot dw$$

$$\frac{dy}{dw} = \frac{\partial y}{\partial x} \cdot \frac{dx}{dw} + \frac{\partial y}{\partial w} \cdot \frac{dw}{dw} = \overbrace{f_x \frac{dx}{dw}}^{\text{indirect}} + \overbrace{f_w}^{\text{direct}}$$

$$\frac{dy}{dw} = \text{total derivative of } y \text{ w.r.t. } w$$

$$\frac{\partial y}{\partial w} = \text{partial derivative of } y \text{ w.r.t. } w$$

$$\begin{aligned} \text{ex. } y &= f(x, w) = 3x - w^2 \\ x &= g(w) = 2w^2 + w + 4, \text{ find } \frac{dy}{dw} \end{aligned}$$

$$\begin{aligned} 1. \text{ By chain rule: } \frac{dy}{dw} &= \frac{d}{dw} 3x - \frac{d}{dw} w^2 \\ &= \frac{d}{dw} (3 \cdot (2w^2 + w + 4)) - 2w \\ &= 12w + 3 - 2w = 10w + 3 \end{aligned}$$

$$\begin{aligned} 2. \text{ Total differential: } dy &= f_x dx + f_w dw \\ &= 3dx + (-2w)dw \end{aligned}$$

$$\frac{dy}{dw} = 3 \frac{dx}{dw} - 2w = 3(4w + 1) - 2w = 10w + 3$$

ex. $Q = Q(K, L, t)$ where $K = K(t)$ and $L = L(t)$

$$\frac{dQ}{dt} = Q_K \frac{dK}{dt} + Q_L \frac{dL}{dt} + Q_t \frac{dt}{dt} \quad \text{Note } \dot{x} = \frac{dx}{dt} \text{ growth rate due to time}$$

$$\dot{Q} = Q_K \cdot \dot{K} + Q_L \cdot \dot{L} + Q_t$$

ex. $y = f(x_1, x_2, u, v)$ where $x_1 = g(u, v)$ and $x_2 = h(u, v)$

$$\frac{dy}{du} \Big|_{v=\bar{v}} = \frac{\partial y}{\partial x_1} \frac{\partial x_1}{\partial u} \Big|_{v=\bar{v}} + \frac{\partial y}{\partial x_2} \frac{\partial x_2}{\partial u} \Big|_{v=\bar{v}} + \frac{\partial y}{\partial u}$$

$$\frac{dy}{dv} \Big|_{u=\bar{u}} = \frac{\partial y}{\partial x_1} \frac{\partial x_1}{\partial v} \Big|_{u=\bar{u}} + \frac{\partial y}{\partial x_2} \frac{\partial x_2}{\partial v} \Big|_{u=\bar{u}} + \frac{\partial y}{\partial v}$$

ex. Utility function $U = U(C, l)$

where C = consumption, l = leisure = $24 - n$, and $C = f(n)$

Given $\frac{\partial U}{\partial C} > 0$ and $\frac{\partial U}{\partial l} > 0$, $U = U(f(n), l(n)) = U(f(n), 24 - n)$, find $\frac{dU}{dn}$

$$\frac{dU}{dn} = U_c \cdot \frac{dC}{dn} + U_l \cdot \frac{dl}{dn} = MU_C \cdot \frac{dC}{dn} + MU_l \cdot \frac{dl}{dn}$$

$$= MU_C \cdot f'(n) + MU_l \cdot (-1) = MU_C \cdot f'(n) - MU_l$$

$$\frac{dU}{dn} > 0 \text{ if } MU_C \cdot f'(n) > MU_l$$

$$\frac{dU}{dn} < 0 \text{ if } MU_C \cdot f'(n) < MU_l$$

4 Implicit Function

Consider $\left\{ \begin{array}{l} y = f(x) = 3x^4 \rightarrow \text{explicit function} \\ F(y, x) = y - 3x^4 \rightarrow \text{implicit function} \end{array} \right\}$ same function in different format

A given function of the form $F(y, x_1, x_2, \dots, x_n) = 0$ defines an implicit function of the form $f(x_1, x_2, \dots, x_n)$ if

1. $F(y, x_1, x_2, \dots, x_n)$ has continuous partial derivatives:

$$\frac{\partial F}{\partial y}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n} \Rightarrow \text{all continuous function}$$

2. $\frac{dF}{dy} \neq 0$

ex. Given $F(y, x, w) = y^3x^2 + w^2 + yxw - 3 = 0$, does $F(y, x, w)$ defines an implicit function $y = f(x, w)$?

Check 2 conditions:

1. continuous partial derivatives $\rightarrow \frac{\partial F}{\partial y}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial w}$

$$\left. \begin{aligned} \frac{\partial F}{\partial y} &= 3y^2x^2 + xw \\ \frac{\partial F}{\partial x} &= 2y^3x + yw \\ \frac{\partial F}{\partial w} &= 2w + xy \end{aligned} \right\} \text{all are continuous?}$$

2. $\frac{dF}{dy} \neq 0$

$$\frac{dF}{dy} = 3y^2x^2 + xw \quad \text{if evaluate at } (y, x, w) = (1, 1, 1) \frac{dy}{dx} = 3(1)^2(1^2) + (1)(1) = 4 \neq 0$$

$\therefore$ implicit function $y = f(x, w)$ is defined by $F(y, x, w)$

Derivatives of implicit functions

$$F = F(y, x_1, x_2, \dots, x_n) \Rightarrow y = f(x_1, x_2, \dots, x_n)$$

$$dF = d(0) = 0 \quad \text{or}$$

$$dF = F_y dy + F_1 dx_1 + \dots + F_n dx_n = 0$$

$$(\text{From } dy = f_1 dx_1 + \dots + f_n dx_n)$$

$$dF = F_y f_1 dx_1 + F_1 dx_1 + F_y f_2 dx_2 + F_2 dx_2 + \dots + F_y f_n dx_n + F_n dx_n$$

$$dF = (F_y f_1 + F_1) dx_1 + \dots + (F_y f_n + F_n) dx_n = 0$$

$$\text{Holds if } F_y f_i + F_i = 0 \forall i = 1, \dots, n$$

$$\therefore f_i = \frac{-F_i}{F_y} \quad \text{or} \quad \frac{\partial f}{\partial x_i} = \frac{-\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}}$$

ex.1 Find $\frac{dy}{dx}$ for implicit function $y - 3x^4 = 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-12x^3}{1} = 12x^3 \quad \text{by implicit function}$$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^4) = 12x^3 \quad \text{by normal differentiation}$$

ex.2 Find $\frac{\partial y}{\partial x}$ for any implicit function that may be defined by

$$F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$$

$$\frac{\partial y}{\partial w} = -\frac{F_w}{F_y} = \frac{-(3w^2 + xy)}{(3x^2y^2 + xw)} \Big|_{(1,1,1)} = -1$$

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = \frac{-(2y^3x + yw)}{(3x^2y^2 + xw)} \Big|_{(1,1,1)} = -\frac{3}{4}$$

ex.3 $F(Q, K, L) = 0$ defines $Q = f(K, L)$ Find MP_K, MP_L

$$\textcircled{1} \quad \frac{\partial Q}{\partial K} = -\frac{F_K}{F_Q} = MP_K$$

$$\frac{\partial Q}{\partial L} = -\frac{F_L}{F_Q} = MP_L$$

$$\frac{MP_L}{MP_K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} \Big|_{Q=\bar{Q}} = \frac{\partial K}{\partial L} = ?$$

$$\textcircled{2} \quad Q = f(K, L) = 0$$

$$dQ = f_K dK + f_L dL = 0$$

$$\frac{dK}{dL} = -\frac{f_L}{f_K} = -\frac{MP_L}{MP_K}$$

ex.4 If $Q(K, L) = 16K^{0.25}L^{0.75} = 2,144$, use implicit function rule to find slope

$$\frac{dK}{dL} = -MRTS$$

$$F(Q, K, L) = 16K^{0.25}L^{0.75} - 2,144 = 0$$

$$\frac{\partial K}{\partial L} = -\frac{F_L}{F_K} = \frac{-16(0.75)K^{0.25}L^{-0.25}}{16(0.25)K^{-0.75}L^{0.75}} = -\frac{3K}{L}$$

(Application: partial Market Equilibrium)

Suppose now Q^d is a function of P as well as Y

$$Q^d = Q^s$$

$$Q^d = D(P, Y)$$

$$Q^s = S(P)$$

where $D_P < 0$, $S_P > 0$, and $D_Y > 0$

$$\text{Endogenous variable} = Q, P$$

$$\text{Exogenous variable} = Y$$

$$\text{Equilibrium condition} : Q^d = Q^s \text{ or } D(P, Y) - S(P) = 0$$

$$\text{Equilibrium price} : P^* = P^*(Y)$$

$$\text{Equilibrium identity} : D(P^*, Y) - S(P^*) \equiv 0$$

where $F(P^*, Y) = D(P^*, Y) - S(P^*)$

Find $\frac{dP^*}{dY}$, $\frac{dQ^*}{dY}$

I.

$$1) \frac{dP^*}{dY_0} = -\frac{F_Y}{F_{P^*}} \text{ by implicit function theorem}$$

$$\Rightarrow F_{Y_0} = \frac{\partial D}{\partial Y_0} \text{ (no } Y \text{ in } S)$$

$$F_P = \frac{\partial D}{\partial P} - \frac{\partial S}{\partial P}$$

$$\therefore \frac{dP^*}{dY_0} = \frac{-\frac{\partial D}{\partial Y_0}}{\frac{\partial D}{\partial P} - \frac{\partial S}{\partial P}} > 0$$

$$2) \frac{dQ^*}{dY_0} = \frac{d}{dY_0}(S(P^*(Y_0))) = \frac{dS}{dP} \frac{dP}{dY} > 0$$

$$\frac{dQ^*}{dY_0} = \frac{d}{dY_0}(D(P^*(Y_0), Y_0)) = \frac{dD}{dP} \cdot \frac{dP}{dY} + \frac{dY_0}{dY_0} = D_P \cdot \frac{dP}{dY} + 1$$

II.

Use total derivative for $\frac{dP^*}{dY}$ from $F = D(P^*, Y) - S(P^*) = 0$, $P^* = P(Y)$

$$\frac{dF}{dY} = \underbrace{\left(\frac{\partial D}{\partial P^*} \cdot \frac{dP^*}{dY} \right)}_{\text{indirect effect of } Y \text{ on } D} + \underbrace{\frac{\partial D}{\partial Y}}_{\text{direct effect of } Y \text{ on } D} - \underbrace{\frac{\partial S}{\partial P^*} \cdot \frac{dP^*}{dY}}_{\text{indirect effect of } Y \text{ on } S}$$

$$\Rightarrow \left(\frac{\partial D}{\partial P^*} - \frac{\partial S}{\partial P^*} \right) \frac{dP^*}{dY} = - \frac{\partial D}{\partial Y}$$

$$\Rightarrow \frac{dP^*}{dY} = \frac{-\frac{\partial D}{\partial Y}}{\frac{\partial D}{\partial P^*} - \frac{\partial S}{\partial P^*}}$$

We can find $\frac{dP^*}{dY}$ with the same process we have in I.

III.

$$Q^d : dQ^* = D_P dP + D_Y dY \quad \Rightarrow \quad \frac{dQ^*}{dY_0} - D_P \frac{dP}{dY_0} = D_Y \frac{dY}{dY}$$

$$Q^s : dQ^* = S_P dP \quad \Rightarrow \quad \frac{dQ^*}{dY} - S_P \frac{dP}{dY} = 0$$

$$\begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix} \begin{bmatrix} \frac{dQ^*}{dY} \\ \frac{dP^*}{dY} \end{bmatrix} = \begin{bmatrix} D_Y \\ 0 \end{bmatrix}$$

By Cramer's rule

$$\frac{dQ^*}{dY} = \frac{\begin{vmatrix} D_Y & -D_P \\ 0 & -S_P \end{vmatrix}}{\begin{vmatrix} 1 & -D_P \\ 1 & -S_P \end{vmatrix}} = \frac{-D_Y S_P}{D_P - S_P} = S_P \cdot \frac{dP^*}{dY}$$

$$\frac{dP^*}{dY} = \frac{\begin{vmatrix} 1 & D_Y \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -D_P \\ 1 & -S_P \end{vmatrix}} = \frac{-D_Y}{D_P - S_P}$$

Interpretation

$$\left. \begin{array}{ll} \text{Normally, } S_P > 0 & \text{slope of supply} \\ D_Y > 0 & \text{normal goods} \\ D_P < 0 & \text{law of demand} \end{array} \right\} \frac{dQ^*}{dY} > 0, \frac{dP^*}{dY} > 0$$

$$\left. \begin{array}{ll} \text{But if } S_P > 0 \\ D_Y < 0 & \text{inferior goods} \\ D_P < 0 \end{array} \right\} \frac{dQ^*}{dY} < 0, \frac{dP^*}{dY} < 0$$

Consider giffen goods

$$\left. \begin{array}{l} S_P > 0 \\ D_Y > 0 \\ D_P > 0 \end{array} \right\} \frac{dQ}{dY} = \frac{-D_Y S_P}{D_P - S_P}$$

Case one, if $S_P > D_P$

$$\frac{dQ}{dY} = \frac{-D_Y S_P}{D_P - S_P} = \frac{\ominus}{\ominus} = \oplus$$

Case two, if $S_P < D_P$

$$\frac{dQ}{dY} = \frac{-D_Y S_P}{D_P - S_P} = \frac{\ominus}{\oplus} = \ominus$$

Example

$$\begin{aligned}
Q^d &= D(P, Y) & \text{endo} &= P, Q \\
Q^s &= S(P, T) & \text{exo} &= Y, T \\
P^* &= P^*(Y, T) \\
Q^* &= Q^*(Y, T)
\end{aligned}$$

Solve for $\frac{dP^*}{dT}$ (suppose $T \not\leftrightarrow Y$)

III.

$$Q^d : dQ = \frac{\partial D}{\partial P} \cdot dP + \frac{\partial D}{\partial Y} \cdot dY \Rightarrow \frac{dQ}{dT} - D_P \frac{dP}{dT} = D_Y \frac{dY}{dT} \Rightarrow \frac{dQ}{dT} - D_P \frac{dP}{dT} = 0$$

$$Q^s : dQ = \frac{\partial S}{\partial P} \cdot dP + \frac{\partial S}{\partial T} \cdot dT \Rightarrow \frac{dQ}{dT} - S_P \frac{dP}{dT} = S_T \frac{dT}{dT} \Rightarrow \frac{dQ}{dT} - S_P \frac{dP}{dT} = 0$$

$$\therefore \begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix} \begin{bmatrix} \frac{dQ}{dT} \\ \frac{dP}{dT} \end{bmatrix} = \begin{bmatrix} 0 \\ S_T \end{bmatrix}$$

$$\frac{dP^*}{dT} = \frac{\begin{vmatrix} 1 & 0 \\ 1 & S_T \end{vmatrix}}{\begin{vmatrix} 1 & -D_P \\ 1 & -S_P \end{vmatrix}} = \frac{S_T}{D_P - S_P}$$

$$S_T < 0$$

$$S_P > 0$$

$$D_P < 0$$

$$\frac{dQ^*}{dT} = \frac{D_P \cdot S_T}{D_P - S_P} < 0$$

How about $\frac{\partial P^*}{\partial Y}, \frac{\partial Q^*}{\partial Y}$?

I.

$$F = D(P, Y) - S(P, T) = 0$$

$$\left. \begin{aligned}
\frac{dP}{dT} &= \frac{-F_T}{F_P} = \frac{-(-S_T)}{D_P - S_P} = \frac{S_T}{D_P - S_P} > 0 \\
\frac{dP}{dY} &= \frac{-F_Y}{F_P} = \frac{-D_Y}{D_P - S_P} > 0
\end{aligned} \right\} \text{How about } \frac{dQ^*}{dT}, \frac{dQ^*}{dP} ?$$

(Application: IS-LM model)

$$\begin{array}{llll}
\text{IS:} & Y & = & C(Y) + I(r) + G_0 & \text{endo} & = & Y, r \\
\text{LM:} & L(Y, r) & = & M_0 & \text{exo} & = & M_0, G_0
\end{array}$$

Solve for $\frac{dr^*}{dM_0}$

1) take total differential

$$\begin{array}{ll}
\text{IS:} & dY = C'(Y)dY + I'(r)dr + dG_0 \\
\text{LM:} & L_y dY + L_r dr = dM_0
\end{array}$$

2) move endogenous variables to left-hand side to form the matrix.

$$\begin{array}{rcl}
dY - C'(Y)dY - I'(r)dr & = & dG_0 \\
L_y dY + L_r dr & = & dM_0
\end{array}$$

3) divide all with dM_0

$$\begin{array}{rcl}
[1 - C'(Y)] \frac{dY}{dM_0} + I'(r) \frac{dr}{dM_0} & = & \frac{dG_0}{dM_0} = 0 \\
L_y \frac{dY}{dM_0} + L_r \frac{dr}{dM_0} & = & \frac{dM_0}{dM_0} = 1
\end{array}$$

4) write matrix form

$$\begin{bmatrix} 1 - C'(Y) & I'(r) \\ L_y & L_r \end{bmatrix} \begin{bmatrix} \frac{dY^*}{dM_0} \\ \frac{dr^*}{dM_0} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

5) solve by Cramer's rule

$$\frac{dr^*}{dM_0} = \frac{\begin{vmatrix} 1 - C'(Y) & 0 \\ L_y & 1 \end{vmatrix}}{\begin{vmatrix} 1 - C'(Y) & I'(r) \\ L_y & L_r \end{vmatrix}} = \frac{1 - C'(Y)}{L_r[1 - C'(Y)] + I'(r)L_y} \Rightarrow \frac{\oplus}{\ominus} < 0$$

How about $\frac{dr^*}{dG_0}$, $\frac{dY^*}{dM_0}$, and $\frac{dY^*}{dG_0}$?