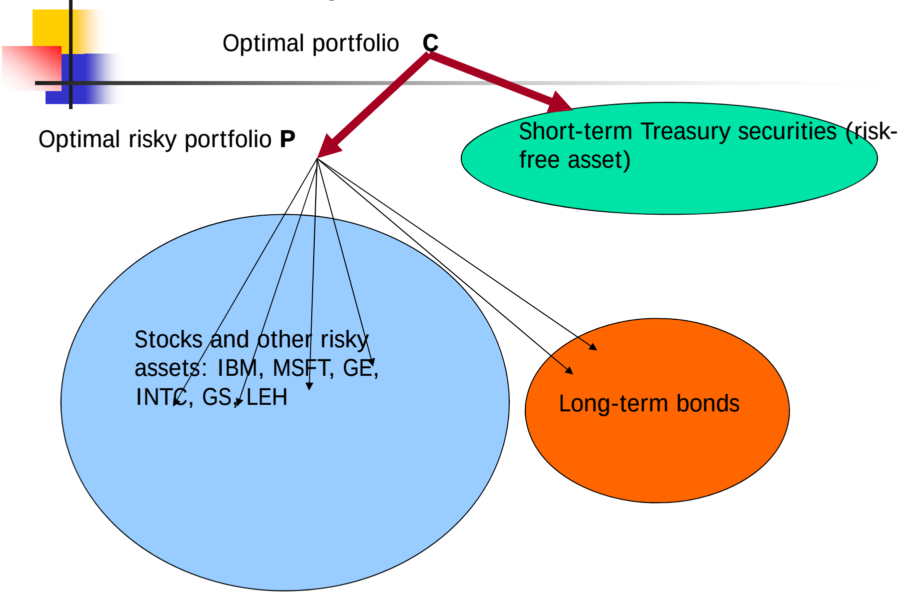
BACHELOR of ECONOMICS
International Program

Thammasat University | Thailand

Portfolio theory



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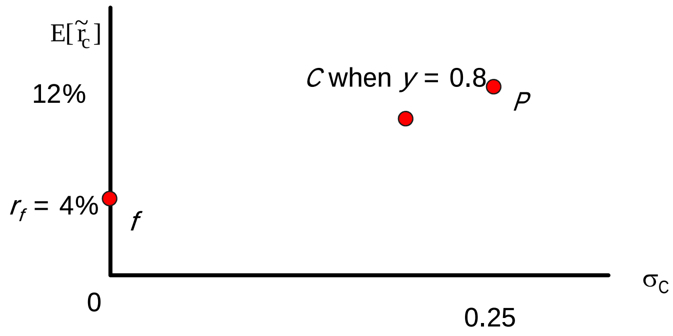
Optimal asset allocation

- Find possible portfolios with different fraction invested in the risky portfolio



Optimal asset allocation

What happens if we choose to put 80% in P and 20% in the riskfree rate. What will be the expected return and standard deviation of this new portfolio C?

$$E(\tilde{r}_C) = yE(\tilde{r}_P) + (1-y)r_f \quad =$$

0.8*0.12+0.2*0.04=0.104

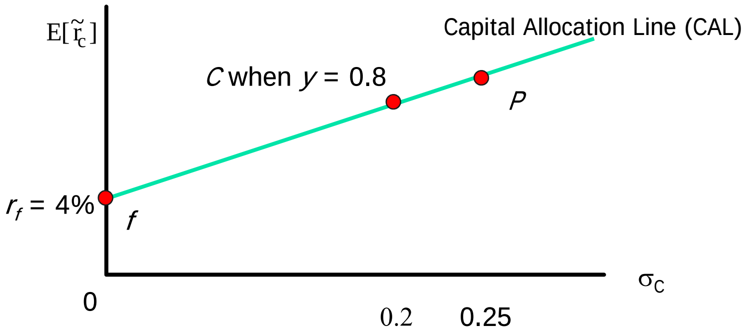
$$\sigma_C^2 = y^2\sigma_P^2 + (1-y)^2\sigma_f^2 + 2y(1-y)\sigma_P\sigma_f\rho_{P,f}$$

$$\sigma_C^2 = y^2\sigma_P^2$$

$$\sigma_C = y\sigma_P$$ =0.25*0.8=0.2

Optimal asset allocation

y	$E[\tilde{r}_c]$	$\sigma[\tilde{r}_c]$
0	0.04	0.0
0.2	0.056	0.05
0.8	0.104	0.20
1.0	0.120	0.25



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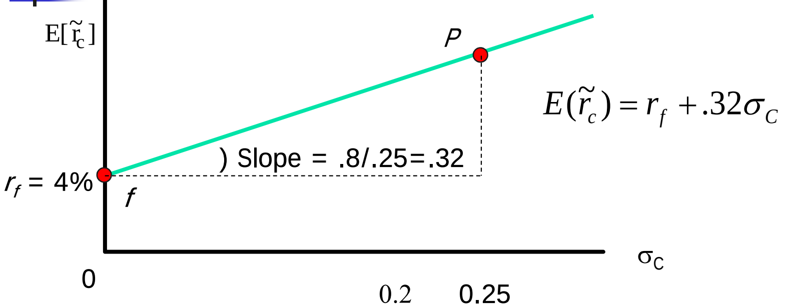
Optimal asset allocation



- Slope of the CAL equals reward-to-variability ratio or the Sharpe ratio

$$\frac{E[\tilde{r}_p]-r_f}{\sigma_p} = \frac{0.08}{.25} = 0.32$$

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Optimal asset allocation

The capital allocation line:

- The CAL represents the investment opportunity set of combinations of the riskfree asset and a risky asset
- The slope of the CAL equals the Sharpe ratio
- When the borrowing and lending rate is the same, all combinations of P and the riskfree asset have the same slope

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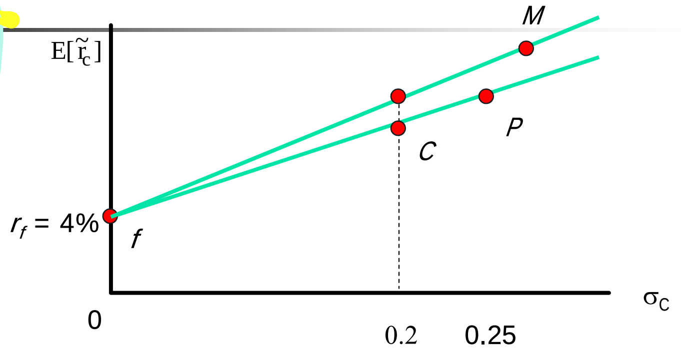
Concept check



• Do you prefer the combination of the riskfree asset and M or P?

G) P
Y) M
R) no difference

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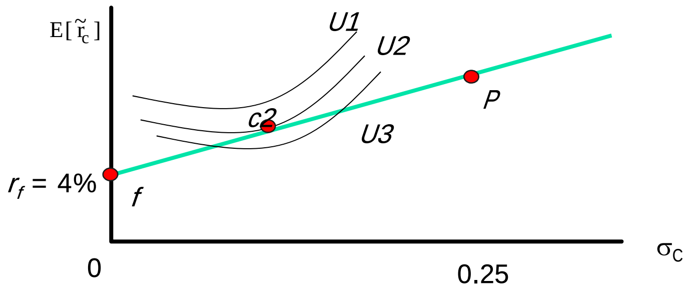
Optimal asset allocation

To choose the optimal y, we maximize utility given all the possible choices of the complete portfolio



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Theoretical derivation of y* (skip)

$$\underset{y}{Max}\ U = E[\tilde{r}_C] - .5A\sigma_C^2$$

$$E[\tilde{r}] = r_f + y[E[\tilde{r}_P] - r_f]$$

$$\sigma^2 = y^2\ \sigma^2_P$$

$$\underset{y}{Max}\ U = r_f + y(E[\tilde{r}_P] - r_f) - .5Ay^2\sigma_P^2$$

$$\frac{dU(y)}{dy} = 0 \rightarrow [E[\tilde{r}_P] - r_f] - Ay^*\sigma_P^2 = 0$$

$$\frac{d^2u}{dy^2} < 0 \rightarrow \text{maximum}$$

$$y^* = \frac{E[\tilde{r}_P] - r_f}{A\sigma_P^2}$$

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Optimal asset allocation

- Let y* denote the optimal y
- y* is at the point where the tangent of the utility indifferent curve equals the slope of the CAL
- $$y^* = \frac{E[r] - r_f}{A\sigma^2}$$
- y* is negatively related to risk aversion, but positively related to the risk premium

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Optimal asset allocation

- Asset allocation for $100,000 between

  1. The riskfree that returns 4%

  2. A diversified portfolio P that has an expected return of 12% and a standard deviation of returns of 25%
- If your risk aversion level is 2 how much do you put in the risky asset?
- Your optimal y is
- Invest $64,000 in P and $36,000 in the riskfree asset

$$y^* = \frac{E[\tilde{r}_p] - r_f}{A \bullet \sigma_p^2} = \frac{.12 - .04}{2 \bullet (0.25)^2} = 0.64$$

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Optimal asset allocation

- It was determined that P has expected return of 12% and standard deviation of 25%.
- What is the expected return and standard deviation of the optimal portfolio C?
- Expected return = (1-0.64)*0.04+0.64*0.12 = 9.12%
- STD = = 0.25*0.64 = 0.16 = 16%

$$\sigma_c = \sigma_P y^*$$

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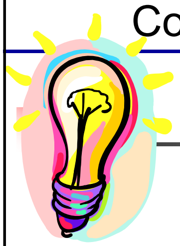
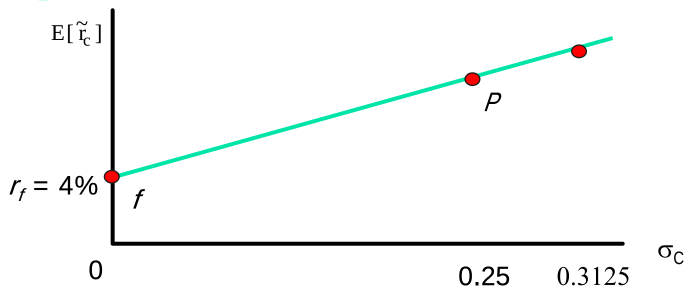
Concept check



y	$E[\tilde{r}_c]$	$\sigma[\tilde{r}_c]$
0.8	0.104	0.20
1.0	0.120	0.25
1.25	0.140	0.3125



What happens when y >1? For example y=1.25

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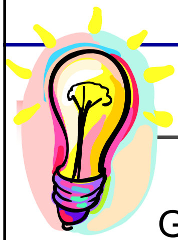

Concept check

G) You made a mistake in the calculations

Y) The optimal investment is to invest 25% of your portfolio in the riskfree asset and 125% in the risky asset

R) The optimal investment is to borrow 25% of the original investment. Invest 125% in the risky asset

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Practice question

- Suppose you are deciding to invest $500,000 between a portfolio of risky assets P that has an expected return of 14% and a standard deviation of 20% and the riskfree asset, which returns 4%. You have determined that your risk aversion level is around 4. How much money should you invest in P and in the riskfree asset?
- What is the expected return and standard deviation of the new portfolio?

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Other ways to find asset allocation:

- Suppose you are deciding to invest $500,000 between a portfolio of risky assets P that has an expected return of 14% and a standard deviation of 20% and the riskfree asset, which returns 4%. You have determined that your risk aversion level is around 4. How much money should you invest in P and in the riskfree asset?
- What is the expected return and standard deviation of the new portfolio?

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Different methods to find asset allocation

Assumption: The risky portfolio being considered is the best in terms of risk-return trade off (optimum risky asset)

1. Use risk aversion level

2. Start with required rate of return and find optimum allocation. Then check if risk is within acceptable level. If too high, need to reduce require rate of return.

3. Start with the maximum rusk tolerable by client. Find optimum allocation. The check if required rate of return is enough. If not need to increase risk.

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Method 2: start with required rate of return

- You have determine that the optimum risky portfolio is P. P has expected rate of return of 12% and standard deviation of returns equals 20%. Your client required a rate of return of 10% a year. How much should she invest in P and how much in risk free which is yielding 3%? What is the standard deviation of your clients portfolio if she invest according to your recommendation?

$$E[\tilde{r}_p] = yE[\tilde{r}_p] + (1-y)E[\tilde{r}_f]$$

$$\sigma[\tilde{r}_p] = \sigma_p y$$

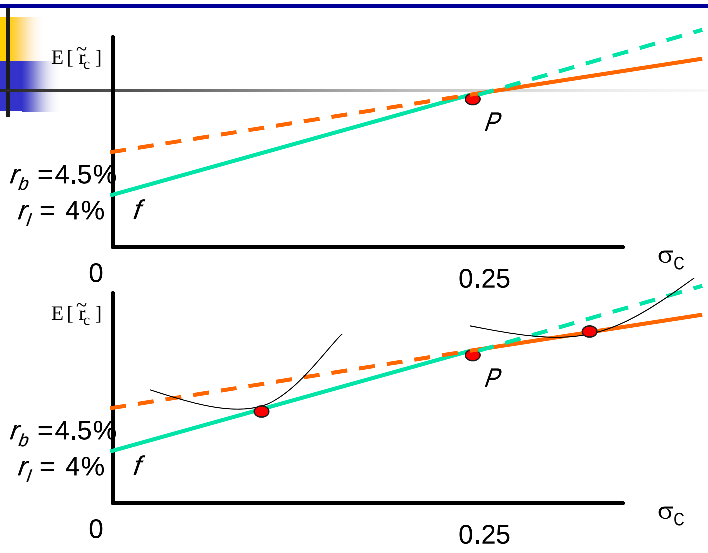
Method 3: start with tolerable risk

- You have determine that the optimum risky portfolio is P. P has expected rate of return of 12% and standard deviation of returns equals 20%. Your client is comfortable investing with standard deviation of his portfolio no more than 15% How much should she invest in P and how much in risk free which is yielding 3%? What is the expected rate of return of your clients portfolio if he invest according to your recommendation?

$$E[\tilde{r}_p] = yE[\tilde{r}_p] + (1-y)E[\tilde{r}_f]$$

$$\sigma[\tilde{r}_p] = \sigma_p y$$

Different borrowing and lending rates



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Method 3: start with tolerable risk

•You have determine that the optimum risky portfolio is P. P has expected rate of return of 12% and standard deviation of returns equals 20%. Your client risk aversion level is 2.

1. The borrowing rate is 6% and the lending rate is 4.5%. What is the optimum proportion your client should invest in the risky asset?
2. The borrowing rate is 3.5% and the lending rate is 2%. What is the optimum proportion your client should invest in the risky asset?
3. The borrowing rate is 4.5% and the lending rate is 2.5%. What is the optimum proportion your client should invest in the risky asset?

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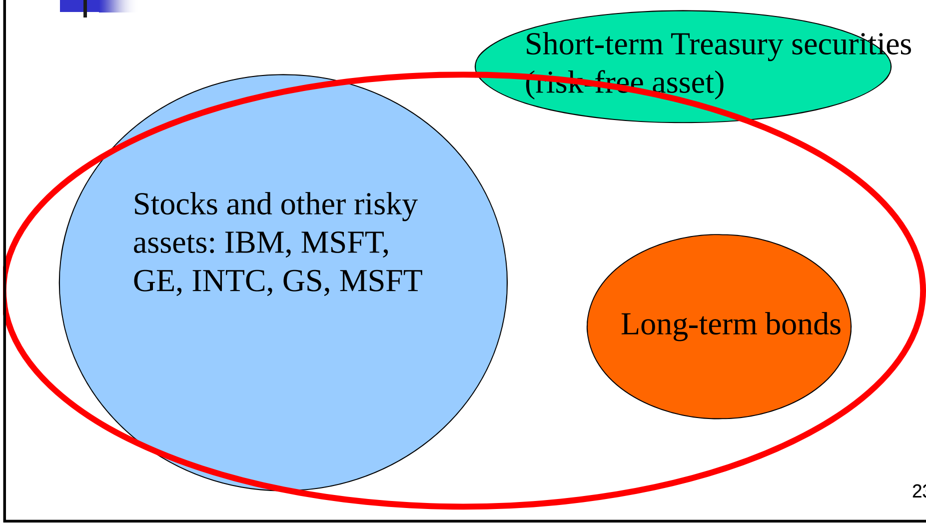
Challenge: How to invest optimally?



Outline

- The interest rate
- Return
- Variance risk
- Investors preference
- Risk-return trade off

Investments

What is the optimal risky portfolio?

Investment Decision

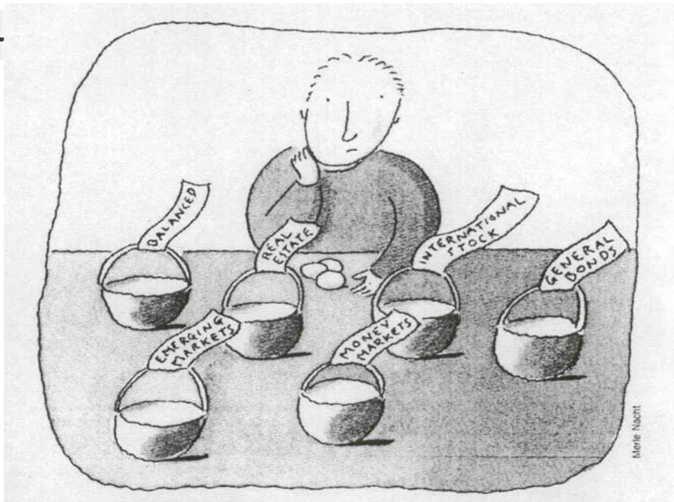
(1-y) — Riskfree asset

y — Optimal risky portfolio, P

w_1 — asset 1

w_2 — asset 2

What to do?



Road map/Key ideas

- Simple diversification
- The effect of correlation on diversification
- Optimal diversification
- Limits to diversification

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Over the period 1962 – 1998 (Annualized):

Portfolio	Average Rate of Return	Standard Deviation
MKT (Equal-weighted)	12.7%	15.1%
IBM	13.4%	22.0%
GE	16.1%	22.0%
EK	11.3%	21.8%
XOM	15.3%	16.8%

- The STD of the market portfolio is lower than most individual stocks

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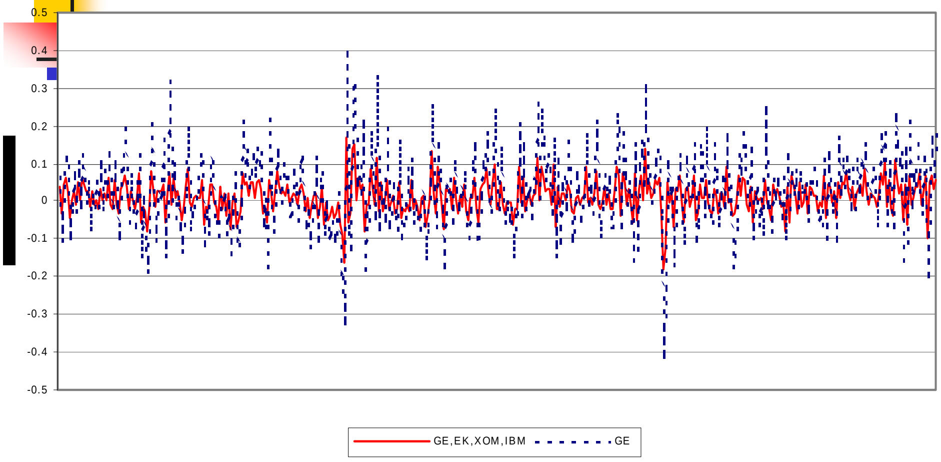
Simple Diversification (1962-1998)



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Simple diversification (1962-1998)

Simple diversification: Equal-weighted portfolio

	IBM	GE	EK	XOM	MKT	IBM,GE	IBM,GE,EK	All 4 stocks
MEAN (per year)	0.134	0.161	0.113	0.153	0.127	0.147	0.136	0.140
STD (per year)	0.227	0.220	0.218	0.168	0.151	0.191	0.175	0.153

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Simple diversification

How diversification works

- Prices of different stocks do not move exactly together
- Variation of a stock price = market-wide variation + firm specific variation.
- Firm-specific variations of different stocks cancel each other out
- STD of a well-diversified portfolio <= STD of an individual stock that make up the portfolio

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Simple diversification

A Well-diversified portfolio should

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Simple diversification

A Well-diversified portfolio should

- Include large number of assets (>40)
- Include assets in different industry/market segment
- Include assets in different countries
- Consider the net current wealth including the present value of future income

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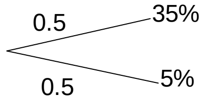
The effect of correlation on diversification

You have $100,000 invested in the S&P, which will return

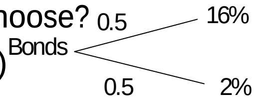


$E[\tilde{r}]=20\%$

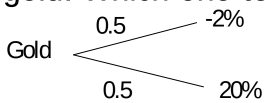
$\sigma=15\%$

Change composition: 70% is invest in the S&P and 30% in either corporate bonds or gold. Which one to choose?

R)



G)



$E[\tilde{r}]=9\%$ $\sigma=0.07$

$E[\tilde{r}]=9\%$ $\sigma=0.11$

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The effect of correlation on diversification

1. 70% S&P + 30% Bonds

0.5 — 35%x0.7+16%x0.3=29.3%

0.5 — 5%x0.7+2%x0.3=4.1%

$E[\tilde{r}]=16.7\%$

$\sigma=12.6\%$

2. 70% S&P + 30% Gold

0.5 — 35%x0.7+(-0.2%)x0.3=23.9%

0.5 — 5%x0.7+20%x0.3=9.5%

$E[\tilde{r}]=16.7\%$

$\sigma=7.2\%\%$

The correlation between two risky securities matters in obtaining a portfolio with lower risk

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The effect of correlation on diversification

Consider a portfolio P that invest w1 in stock 1 and w2 in stock 2. The correlation between the two stocks is ρ

- Expected return $E[\tilde{r}_P]=w_1E[\tilde{r}_1]+w_2E[\tilde{r}_2]$
- Variance $\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2$
- The bordered variance-covariance matrix

	w1	w2
w1	σ_1^2	$\rho\sigma_1\sigma_2$
w2	$\rho\sigma_1\sigma_2$	σ_2^2

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The effect of correlation on diversification

The bordered variance-covariance matrix for 3 assets

	w1	w2	w3
w1	σ_1^2	$\rho_{12}\sigma_1\sigma_2$	$\rho_{13}\sigma_1\sigma_3$
w2	$\rho_{12}\sigma_1\sigma_2$	σ_2^2	$\rho_{23}\sigma_2\sigma_3$
w3	$\rho_{13}\sigma_1\sigma_3$	$\rho_{23}\sigma_2\sigma_3$	σ_3^2

The effect of correlation on diversification

Consider two risky assets: stock 1 and stock 2

- Stock 1: $E[\tilde{r}_1] = 8\% = \mu_1, \sigma[\tilde{r}_1] = \sigma_1 = 12\%$
- Stock 2: $E[\tilde{r}_2] = 13\% = \mu_2, \sigma[\tilde{r}_2] = \sigma_2 = 20\%$
- Riskfree rate is 0.5%

$$E[\tilde{r}_P] = w_1 E[\tilde{r}_1] + w_2 E[\tilde{r}_2]$$

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2$$

The effect of correlation on diversification

When $\rho=1$

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2,$$

$$= (w_1\sigma_1 + w_2\sigma_2)^2$$

$$\sigma_p = w_1\sigma_1 + w_2\sigma_2$$

When the correlation equals 1 the STD of the portfolio equals the weighted average of the STD of the stocks that make up the portfolio. There is no diversification benefit.

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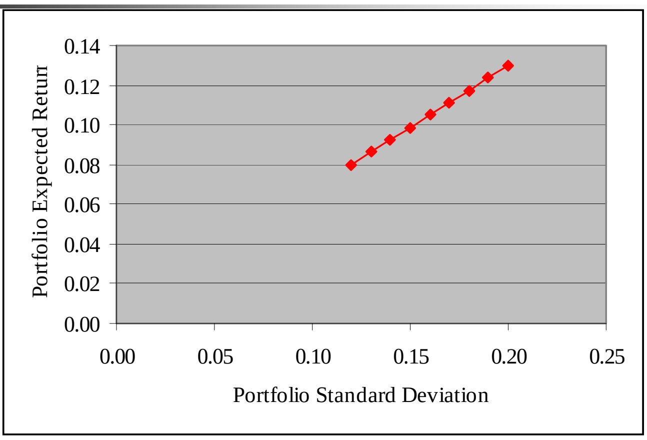
The effect of correlation on diversification

Case: $\rho=1$ there is no diversification benefit



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The effect of correlation on diversification

- When $\rho < 1$ there is always potential for diversification benefit

$$\sigma_P = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2} < w_1\sigma_1 + w_2\sigma_2$$

- Assets with $\rho < 0$ are often referred to as hedged assets.

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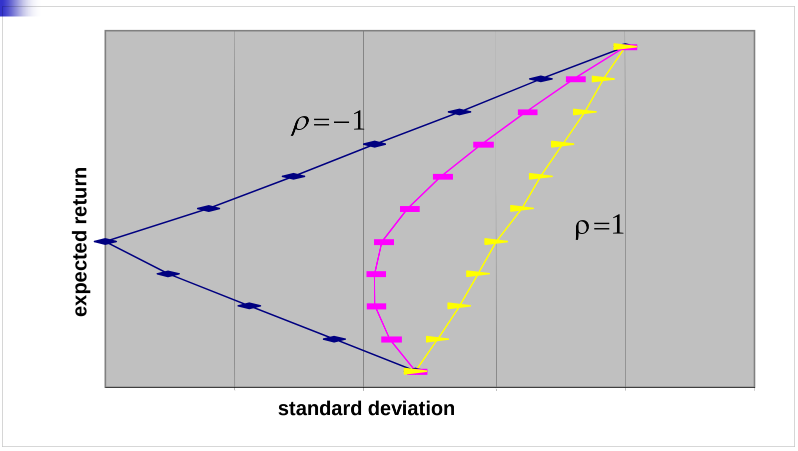
The effect of correlation on diversification



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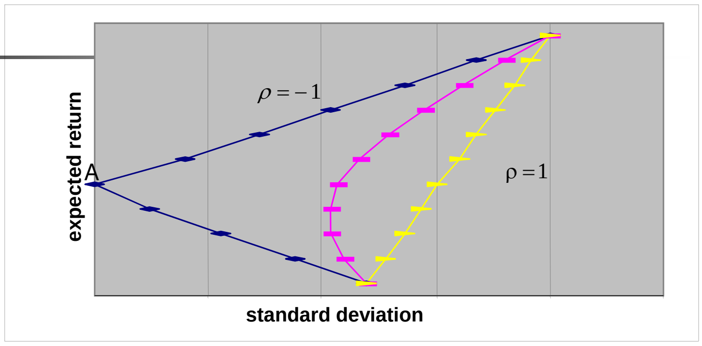
Concept check



What should the expected return at point A be?

G) Zero

Y) Riskfree rate

R) The return on the market portfolio

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The effect of correlation on diversification

Case: $\rho = -1$

$$E[\tilde{r}_p] = w_1 E[\tilde{r}_1] + w_2 E[\tilde{r}_2]$$

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 - 2 w_1 w_2 \sigma_1 \sigma_2$$

$$= (w_1 \sigma_1 - w_2 \sigma_2)^2$$

$$\sigma_p = w_1 \sigma_1 - w_2 \sigma_2$$

To find the point where volatility is zero

$$0 = w_1 \sigma_1 - (1 - w_1) \sigma_2$$

$$w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

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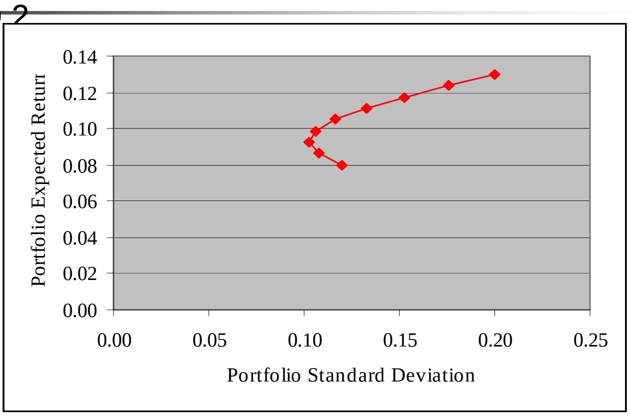
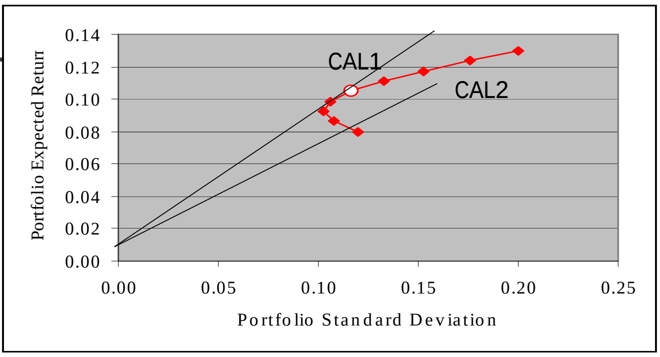
Optimal risky portfolio

Investment opportunity set of two risky assets when ρ =0.2

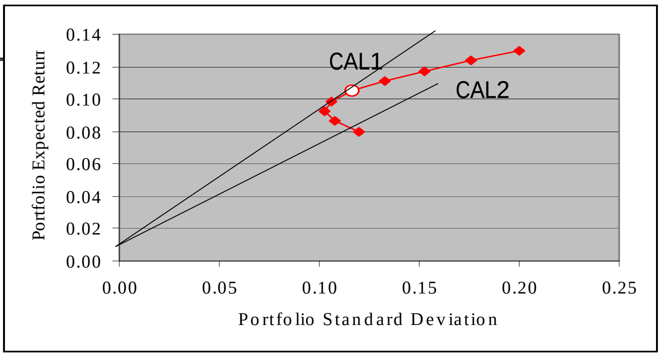


Optimal risky portfolio (visual)



- Adding the riskfree asset generates a number of CAL choices.
- Which CAL is better?

Optimal risky portfolio (visual)



- Adding the riskfree asset generates a number of CAL choices.
- Which CAL is better?
- Choose the CAL with the highest slope (Sharpe ratio)

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Optimal risky portfolio (algebra)

Maximize $(E[\tilde{r}_p]-r_f)/\sigma_p$

w_1, w_2

Subject to: $w_1 = (1-w_2)$

$$E[\tilde{r}_P] = w_1 E[\tilde{r}_1] + w_2 E[\tilde{r}_2]$$

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2$$

We obtain equation 8.7 in the book

$$w_1 = \frac{[E(r_1)-r_f]\sigma_2^2 - [E(r_2)-r_f]Cov(r_1,r_2)}{[E(r_1)-r_f]\sigma_2^2 + [E(r_2)-r_f]\sigma_1^2 - [E(r_1)-r_f+E(r_2)-r_f]Cov(r_1,r_2)}$$

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Optimal risky portfolio

Example: Investing $100,000, riskfree=0.5%

- Stock s 1 and 2: $E[\tilde{r}_1] = 8\%$, $\sigma_1 = 12\%$ $E[\tilde{r}_2] = 13\%$, $= \sigma_2 = 20\%$
- The correlation between the two stock returns is 0

2. Solve for P

$$w_1 = \frac{[E(r_1) - r_f]\sigma_2^2 - [E(r_2) - r_f]Cov(r_1, r_2)}{[E(r_1) - r_f]\sigma_2^2 + [E(r_2) - r_f]\sigma_1^2 - [E(r_1) - r_f + E(r_2) - r_f]Cov(r_1, r_2)}$$

$$w_1 = \frac{[0.08 - 0.005]0.2^2}{[0.08 - 0.005]0.2^2 + [0.13 - 0.005]0.12^2} = 0.625$$

$$w_2 = 0.375$$

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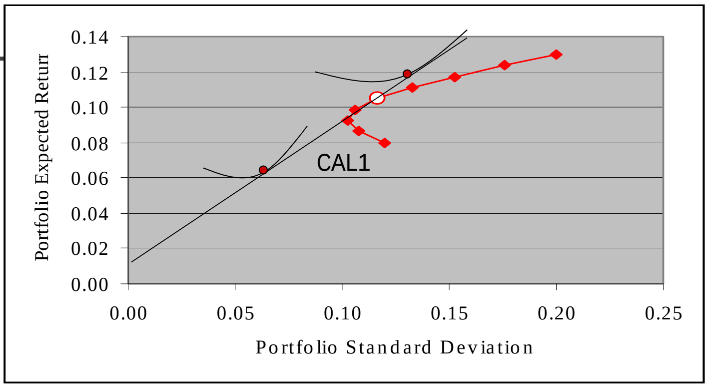
Optimal risky portfolio



- Allocate investment between riskfree asset and P using each investor's utility function $y^* = (E[\tilde{r}_p] - r_f)/(A \bullet \sigma_p^2)$

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Optimal risky portfolio

Investing \$100,000, A=8, riskfree=0.5%

1. Stock s 1 and 2: $E[\tilde{r}_1] = 8\%$, $\sigma_1 = 12\%$ $E[\tilde{r}_2] = 13\%$, $= \sigma_2 = 20\%$

- The correlation between the two stock returns is 0

2. We have w_1 = 0.625 and w_2 = 0.375

$$E[\tilde{r}_P] = 0.625x0.08 + 0.375x0.13 = 0.09875$$

$$\sigma_P = \sqrt{0.625^2 0.12^2 + 0.375^2 0.20^2 + 2x0.625x0.375x0x0.12x0.20} = 0.106$$

3. y*=(0.09875-0.005)/(8x0.106^2) = 1.04

Borrow 4,000; Invest 0.625*1.04*100,000= 65,000 in stock 1; and 0.375*1.04*100,000=39,000 in stock 2

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Concept check

Because diversification is so important, there is never a time when it is optimal to invest more than 90% of a portfolio in one specific asset. Is this true or false?

G) True
R) False

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Example: Investing $100,000, riskfree=0.5%

- Stock 1: $E[\tilde{r}_1] = 8\%\,,\ \sigma_1 = 12\%$
- Stock 2: $E[\tilde{r}_2] = 13\%\,,\ \sigma_2 = 3\%$
- The correlation between the two stock returns is 0

2. Solve for P

$$w_1 = \frac{[E(r_1) - r_f]\sigma_2^2 - [E(r_2) - r_f]Cov(r_1, r_2)}{[E(r_1) - r_f]\sigma_2^2 + [E(r_2) - r_f]\sigma_1^2 - [E(r_1) - r_f + E(r_2) - r_f]Cov(r_1, r_2)}$$

$$w_1 = \frac{[0.08 - 0.005]0.03^2}{[0.08 - 0.005]0.03^2 + [0.13 - 0.005]0.12^2} = 0.036$$

$$w_2 = 0.964$$

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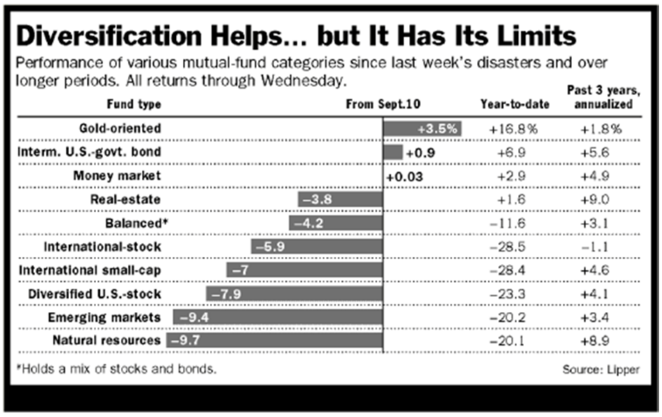
Limits to Diversification

- Date 09/21/2001



Diversification Helps… but It Has Its Limits

Performance of various mutual-fund categories since last week's disasters and over longer periods. All returns through Wednesday.

Fund type	From Sept.10	Year-to-date	Past 3 years, annualized
Gold-oriented	+3.5%	+16.8%	+1.8%
Interm. U.S.-govt. bond	+0.9	+6.9	+5.6
Money market	+0.03	+2.9	+4.9
Real-estate	−3.8	+1.6	+9.0
Balanced*	−4.2	−11.6	+3.1
International-stock	−5.9	−28.5	−1.1
International small-cap	−7	−28.4	+4.6
Diversified U.S.-stock	−7.9	−23.3	+4.1
Emerging markets	−9.4	−20.2	+3.4
Natural resources	−9.7	−20.1	+8.9

*Holds a mix of stocks and bonds.

Source: Lipper

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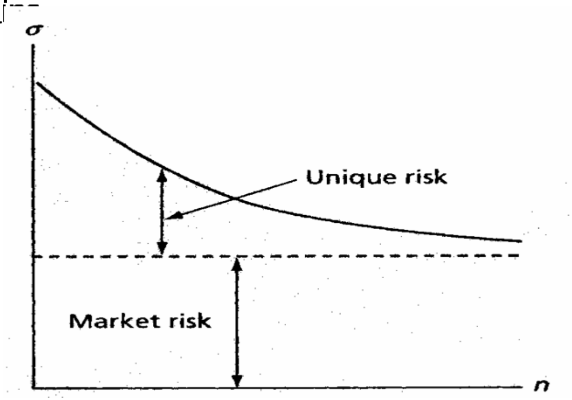
Limits to Diversification

- Total variation of a stock = market risk + unique risk
- Unique risk is diversified away, but market risk remai...



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Summary

- Simple diversification
- The effect of correlation on diversification
  - Portfolio of less that perfectly correlated assets always offer better risk-return opportunities than the individual component securities on their own
- Optimal diversification with two assets
  - Find the investment opportunity set. Pick the best CAL line.
- Limits to diversification
  - Firm specific risk is diversified away, but market risk remains

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