Matrix inverses

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse of** A .

example $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$; $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ $AC=$ $CA=$

The inverse of A is usually denoted by A^{-1}.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all $n \times n$ ***matrices are invertible.*** A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

Fact 1 If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_\_)\_\_\_\_\_\_ = I\_\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.■

Fact 2 The inverse of A^{-1} is A itself.

Fact 3 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$.
Then

_____ $A\mathbf{x} =$ _____ $\mathbf{b}$ and so

$I\mathbf{x} =$ ________ or $\mathbf{x} =$ ________.

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

_____ $A\mathbf{w} =$ _____ $\mathbf{b}$ which means $\mathbf{w} = A^{-1}\mathbf{b}$.

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Fact 4 If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}.$$

Solution: Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Properties of Inverses

Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\_\_\_\_\_\_\_\_)A^{-1}$$

$$= A(\_\_\_\_\_\_\_\_\_)A^{-1} = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Proof part c

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Matrix inversion algorithm

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

➔Taken a column at a time, that equation determines the columns of A^{-1}

A times column j of A^{-1} = column j of I

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$$

Carry out elimination on *all systems simultaneously*.

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A & I \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I & A^{-1} \end{bmatrix}$$

The Gauss-Jordan method

Matrix inversion algorithm

Place A and I side-by-side to form an augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. Then perform row operations on this matrix (which will produce identical operations on A and I). So by Theorem 7:

$\begin{bmatrix} A & I \end{bmatrix}$ will row reduce to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists

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Elementary Matrices

Definition

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

EXAMPLE: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operations on I_m.

Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

For example,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\boxed{E_3E_2E_1A = I_3}.$$

Then multiplying on the right by A^{-1}, we get

$$E_3E_2E_1A\_\_\_\_\_\_ = I_3\_\_\_\_\_\_.$$

So

$$\boxed{E_3E_2E_1I_3 = A^{-1}}$$

The elementary row operations that row reduce A to I$_n$ are the same elementary row operations that transform I$_n$ into A^{-1}.

Matrix Factorisations

A ***factorisation*** of a matrix A is an equations that expresses A as a product of 2 or more matrices.

Gauss elimination can be used to find such factorisations.

$$Ax=b \quad \Rightarrow \quad Ux=c$$

How to relate A to U?

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

E.g.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \quad \Rightarrow \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

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3 elimination steps

i) Subtract 2 times the first equation from the second;
ii) Subtract –1 times the first equation from the third;
iii) Subtract –1 times the second equation from the third

Each step corresponds to an elementary matrix.
Let matrix E for step i), F for step ii) and G for step iii)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \; ; \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$GFEA = U$$

The single matrix that take A to U is GFE

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$GFEA = U$$

$$A = (GFE)^{-1}U = E^{-1}F^{-1}G^{-1}U$$

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = L$$

L= lower triangular

$A = LU$ →Triagular (LU) factorisation

The entries below the diagonal are exactly the multipliers l=2, -1 and -1.

If no row exchanges are required, the original matrix *A* can be written as a product ***A=LU.*** The matrix *L* is lower triangular, with 1's on the diagonal and the multipliers l_{ij} (taken from elimination) below the diagonal.
U is the upper triangular matrix which appears after forward elimination and before back substitution; its diagonal entries are pivots.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \text{ with } L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow$$

Needs a row exchange
cannot be factored into *A=LU*

Remark You have to be careful with *L*. Suppose elimination substracts row 1 from row 2, creating l_{21}. Then suppose it exchanges rows 2 and 3. If that exchange is done in avance, the multiplier will change to l_{31} in *PA=LU*

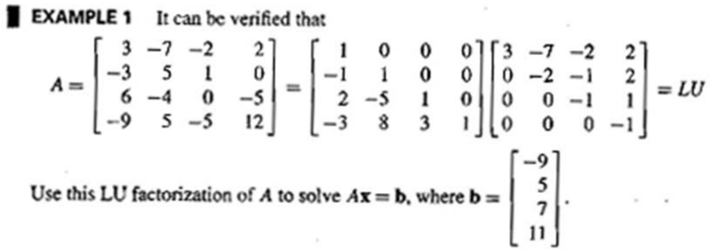
E.g.

$$A=\begin{bmatrix}1&1&1\\1&1&3\\2&5&8\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&0&2\\0&3&6\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&3&6\\0&0&2\end{bmatrix}=U$$

With rows exchanged, we recover LU –but now $l_{31}=1$ and $l_{21}=2$:

$$P=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\text{ and }L=\begin{bmatrix}1&0&0\\2&1&0\\1&0&1\end{bmatrix}$$

$$\underset{m\times n}{A}=\underbrace{\begin{bmatrix}1&0&0&0\\ *&1&0&0\\ *&*&1&0\\ *&*&*&1\end{bmatrix}}_{\substack{L\\ m\times m}}\underbrace{\begin{bmatrix}\bullet&*&*&*&*\\0&\bullet&*&*&*\\0&0&0&\bullet&*\\0&0&0&0&0\end{bmatrix}}_{\substack{U\\ m\times n}}$$



EXAMPLE 1 It can be verified that

$$A=\begin{bmatrix}3&-7&-2&2\\-3&5&1&0\\6&-4&0&-5\\-9&5&-5&12\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\2&-5&1&0\\-3&8&3&1\end{bmatrix}\begin{bmatrix}3&-7&-2&2\\0&-2&-1&2\\0&0&-1&1\\0&0&0&-1\end{bmatrix}=LU$$

Use this LU factorization of A to solve $A\mathbf{x}=\mathbf{b}$, where $\mathbf{b}=\begin{bmatrix}-9\\5\\7\\11\end{bmatrix}$.

$$[L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [I \quad \mathbf{y}]$$

$$[U \quad \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

Algorithm for an LU factorization

1. Reduce **A** to an echelon form **U** by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces **L** to **I**

EXAMPLE 2 Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

$$\begin{array}{cccc} \begin{bmatrix}2\\-4\\2\\-6\end{bmatrix} & \begin{bmatrix}3\\-9\\12\end{bmatrix} & \begin{bmatrix}2\\4\end{bmatrix} & \begin{bmatrix}5\end{bmatrix} \\ \div 2 & \div 3 & \div 2 & \div 5 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix}1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1\end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 1 & -3 & 1 & 0\\ -3 & 4 & 2 & 1\end{bmatrix}$$

An easy calculation verifies that this L and U satisfy $LU = A$.