Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

Full name Jaruvit Chantana **Student ID.** 6104641482

Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1 - 1.5B + 0.9B^2)R_t = 0.25 + \varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean $(\mu) = 0$ and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

reverse characteristic eq : $(x^2 - 1.5x + 0.9)$

$$\lambda_i = \frac{-(-1.5) \pm \sqrt{(-1.5)^2 - 4(1)(0.9)}}{2(1)}$$

Re(polyroot(c(0.9,-1.5,1)))
[1] 0.75 0.75

Im(polyroot(c(0.9,-1.5,1)))
[1] 0.581 -0.581

$\lambda_i = a + bi$

$\lambda_1 = 0.75 + 0.581(i)$ $\quad R = \sqrt{a^2 + b^2}$

$\lambda_2 = 0.75 - 0.581(i)$ $\quad R = \sqrt{(0.75)^2 + (0.581)^2}$

$= 0.949 < 1$

$\therefore$ the model is weak stationarity #

Question 1.2 (10 points) | Your score......................

$$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$$

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

$$R_t - 1.5BR_t + 0.9B^2R_t = 0.25 + a_t$$

$$\Rightarrow R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25 + a_t$$

$$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + a_t$$

Unconditional mean

take $E[\cdot]$ to both side

$$E[R_t] = E[0.25] + E[1.5R_{t-1}] - E[0.9R_{t-2}] + E[a_t]$$

weak stationarity $\Rightarrow$ $E[r_t] = E[r_{t-1}] = E[r_{t-2}] = \ldots = \mu$ = constant

$$E[R_t] - 1.5E[R_t] + 0.9E[R_t] = E[0.25] + E[a_t]$$

$$(1-1.5+0.9)E[R_t] = 0.25 + 0$$

$$E[R_t] = \frac{0.25}{1-1.5+0.9} = 0.625 \ \#$$

Conditional mean

take $E[\cdot|F_{t-1}]$ to both side

$$E[R_t|F_{t-1}] = E[0.25|F_{t-1}] + E[1.5R_{t-1}|F_{t-1}] - E[0.9R_{t-2}|F_{t-1}] + E[a_t|F_{t-1}]$$

$$= 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + 0 \ \#$$

Question 1.3 (10 points) Your score......................

$$R_t - 1.5BR_t + 0.9B^2R_t = 0.25 + a_t$$
$$\Rightarrow R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25 + a_t$$
$$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + a_t$$

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

Deviation form $(R_t - \mu) = 1.5(R_{t-1}-\mu) - 0.9(R_{t-2}-\mu) + a_t$

Unconditional variance: $E[(r_t-\mu)^2]$

take square to both side

$$(R_t-\mu)^2 = 1.5^2(R_{t-1}-\mu)^2 + (-0.9)^2(R_{t-2}-\mu)^2 + a_t^2 + 2(1.5)(-0.9)(r_{t-1}-\mu)(r_{t-2}-\mu)$$
$$+ 2(1.5)(r_{t-1}-\mu)(a_t) + 2(-0.9)(r_{t-2}-\mu)(a_t)$$

take $E(\cdot)$ to both side

$$E[(R_t-\mu)^2] = 1.5^2 E[(R_{t-1}-\mu)^2] + (-0.9)^2 E[(R_{t-2}-\mu)^2] + E[a_t^2]$$
$$+ 2(1.5)(-0.9)E[(r_{t-1}-\mu)(r_{t-2}-\mu)]$$
$$+ 2(1.5)E[(r_{t-1}-\mu)(a_t)] + 2(-0.9)E[(r_{t-2}-\mu)(a_t)]$$

$$var(R_t) = (1.5)^2 var(R_{t-1}) + (-0.9)^2 var(R_{t-2}) + \sigma^2 + (-2.7)cov(r_{t-1}, r_{t-2})$$
$$+ 3\underbrace{cov(r_{t-1}, a_t)}_{0} + (-1.8)\underbrace{cov(r_{t-2}, a_t)}_{0}$$

$$= 2.25\, var(R_{t-1}) + 0.81\, var(R_{t-2}) + \sigma^2 - 2.7\gamma_{2-1}$$

weak stationarity $\rightarrow$ $var(R_t) = var(R_{t-1}) = \dots = constant$

$$var(R_t) - 2.25\,var(R_t) - 0.81\,var(R_t) = \sigma^2 - 2.7\gamma_1$$
$$(1 - 2.25 - 0.81)\,var(R_t) = \sigma^2 - 2.7\gamma_1$$
$$var(R_t) = \frac{0.25 - 2.7\gamma_1}{-2.06}$$

Solve for γ_1

$$cov(r_t, r_{t-j}) = E[(r_t-\mu)(r_{t-j}-\mu)] = \gamma_j$$

take $(r_{t-j}-\mu)$ and $E[\cdot]$ to both side of AR(2)

$$\underbrace{E[(r_t-\mu)(r_{t-j}-\mu)]}_{\gamma_j} = 1.5\underbrace{E[(r_{t-1}-\mu)(r_{t-j}-\mu)]}_{} \Rightarrow \gamma_{j-1}$$
$$+ (-0.9)\underbrace{E[(r_{t-2}-\mu)(r_{t-j}-\mu)]}_{} \Rightarrow \gamma_{j-2}$$
$$+ \cancel{E[a_t(r_{t-j}-\mu)]}\ 0$$

$$\gamma_j = 1.5\gamma_{j-1} - 0.9\gamma_{j-2}$$
$$\Rightarrow \gamma_1 = 1.5\gamma_0 - 0.9\gamma_1$$
$$1.9\gamma_1 = 1.5\, var(r_t)$$

$$\gamma_1 = 0.789 \operatorname{var}(r_t)$$

back to $\operatorname{var}(r_t) = \dfrac{0.25 - 2.7\gamma_1}{-2.06}$

$$= \frac{0.25 - 2.7(0.789 \operatorname{var}(r_t))}{-2.06}$$

$$-2.06 \operatorname{var}(r_t) = 0.25 - 2.1303 \operatorname{var}(r_t)$$

$$0.0703 \operatorname{var}(r_t) = 0.25$$

$$\operatorname{var}(r_t) = \frac{0.25}{0.0703} = 3.556 \quad \#$$

Conditional variance

take $\operatorname{var}(\cdot \mid F_{t-1})$ to both side

$$\operatorname{var}(R_t \mid F_{t-1}) = \cancel{\operatorname{var}(0.25 \mid F_{t-1})}\ 0 + \cancel{(1.5)^2 \operatorname{var}(R_{t-1} \mid F_{t-1})}\ 0$$

$$+ (-0.9)^2 \cancel{\operatorname{var}(R_{t-2} \mid F_{t-1})}\ 0$$

$$+ \cancel{\operatorname{var}(a_t \mid F_{t-1})}\ \sigma_a^2$$

$$+ 2(1.5)(-0.9) \cancel{\operatorname{cov}(R_{t-1}, R_{t-2} \mid \cdot)}\ \gamma_1$$

$$+ \cancel{2(1.5)\operatorname{cov}(R_{t-1}, a_t \mid \cdot)}\ 0$$

$$+ \cancel{2(-0.9)\operatorname{cov}(R_{t-2}, a_t \mid \cdot)}\ 0$$

$$\operatorname{var}(R_t \mid F_{t-1}) = \sigma_a^2 + \gamma_1$$

$$= \sigma_a^2 + 0.789 \operatorname{var}(R_t \mid F_{t-1})$$

$$\operatorname{var}(R_t \mid F_{t-1}) = \frac{0.25}{0.211} = 1.185 \quad \#$$

Question 1.4 (10 points)

Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

from $\gamma_j = 1.5\gamma_{j-1} - 0.9\gamma_{j-2}$

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{1.5\gamma_{j-1}}{\gamma_0} - \frac{0.9\gamma_{j-2}}{\gamma_0}$$

$$\rho_j = 1.5\rho_{j-1} - 0.9\rho_{j-2}$$

$\rho_1 = 1.5\rho_0 - 0.9\rho_1$

$1.9\rho_1 = 1.5(1)$

$\rho_1 = \frac{1.5}{1.9} = 0.789$

$\rho_2 = 1.5\rho_1 - 0.9\rho_0$

$\rho_2 = 1.5(0.789) - 0.9(1)$

$\rho_2 = 0.2835$

Question 1.5 (10 points) Your score......................

$R_t = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + a_t$

$R_{1001} = 0.25 + 1.5 R_{1000} - 0.9 R_{999} + a_{1001}$

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

1- step ahead forecasting

point estimator :

$$\hat{R}_{1000}(1) = 0.25 + 1.5 R_{1000} + (-0.9) R_{999}$$
$$= 0.25 + 1.5(0.01) + (-0.9)(0.02)$$
$$= 0.25 + 0.015 - 0.018$$
$$= 0.247$$

the 1-step ahead forecast error :

$$R_{1001} - \hat{R}_{1000}(1) = e_{1000}(1) = a_{1001}$$

variance of forecasting for 1-step ahead forecasting

$$var(e_{1000}(1) \mid \cdot) = var(a_{1001} \mid \cdot)$$
$$= \sigma_a^2$$
$$= 0.25$$

conduct the interval forecasting

$$\hat{R}_{1000}(1) \pm Z_{\frac{\alpha}{2}} \cdot \sqrt{var(a_{1001} \mid \cdot)}$$

$$\simeq 0.247 \pm 1.96 \cdot 0.5$$

$$\simeq 0.247 \pm 0.98$$

$$-0.733 \leq R_{1001} \leq 1.227 \quad \text{at } 95\% \text{ CI}$$

$R_t = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + a_t$

2- step ahead forecasting

point estimator

$$\hat{R}_{1000}(2) = 0.25 + 1.5\ \hat{R}_{1000}(1) + (-0.9)\ R_{1000}$$
$$= 0.25 + 1.5(0.247) + (-0.9)(0.01)$$
$$= 0.25 + 0.3705 - 0.009$$
$$= 0.6115$$

variance

$$e_{1000}(2) = R_{1002} - \hat{R}_{1000}(2) = 1.5\, a_{1001} + a_{1002}$$

compute the variance

$$\mathrm{var}(e_{1000}(2)|\cdot) = \mathrm{var}(1.5\, a_{1001} + a_{1002}|\cdot) \qquad a \overset{iid}{\sim} N(0, \sigma_a^2)$$
$$= (1.5)^2\ \mathrm{var}(a_{1001}|\cdot) + \mathrm{var}(a_{1002}|\cdot) + \underbrace{\cancel{2\,\mathrm{cov}(1.5\, a_{1001}, a_{1002}|\cdot)}}_{0}$$
$$= (1.5)^2 \cdot \sigma_a^2 + \sigma_a^2$$
$$= 2.25 \cdot 0.25 + 0.25$$
$$= 0.8125$$

interval forecasting of 2-step ahead forecasting

$$\hat{R}_{1000}(2) \pm 1.96\sqrt{0.8125}$$
$$0.6115 \pm 1.767$$

$$-1.1555 \leq R_{1002} \leq 2.3785 \quad \text{at } 95\%\ \text{CI.}$$

for ∞-step ahead forecasting

in the long run the return is going to approach means as well as variance so

$$E(R_t) - 1.96\sqrt{\mathrm{var}(R_t)} \leq R_t \leq 0.01 + 1.96\sqrt{0.25}$$
$$-0.97 \leq R_t \leq 0.99$$