EE320 Chapter 8
Optimization without Constraint: More-Than-One-Independent-Variable Case

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1 Review

Recall from one-independent variable optimizations without constraints:

$$\begin{aligned} y &= f(x) \\ dy &= f'(x)dx \end{aligned}$$

FOC for an optimum necessary condition

$\frac{dy}{dx} = f'(x) = 0$ or first-order differential condition for any dx $\neq 0$ and $dy = 0$

SOC to check sufficient condition

$$\begin{aligned} \frac{d^2y}{dx^2} = f''(x) &< 0 \text{ maximum} \\ f''(x) &> 0 \text{ minimum} \\ f''(x) &< 0 \text{ indetermined (max,min,inflection point)} \\ \Rightarrow d^2y &= d(dy) = d(f'(x)dx) = (df'(x))dx = f''(x)dx \end{aligned}$$

2 Two-choice-variable optimization

Suppose that we have a function with 2 choice variables: $z = f(x, y)$

First-order-necessary condition is :

$$\begin{aligned} dz &= f_x dx + f_y dy = 0 \\ \therefore dz &= 0 \text{ when } f_x = f_y = 0 \end{aligned}$$

Theorem: A differentiable function $z = f(x, y)$ can only have a maximum or minimum at an interior point (x_0, y_0) if it is a stationary point. That is , if the point $(x.y) = (x^*, y^*)$ satisfies the two FOC equations:

$$f_x(x_0^*, y_0^*) = 0$$
$$f_y(x_0^*, y_0^*) = 0$$

ex. $z = f(x, y) = 10x + 10y + xy - x^2 - y^2$

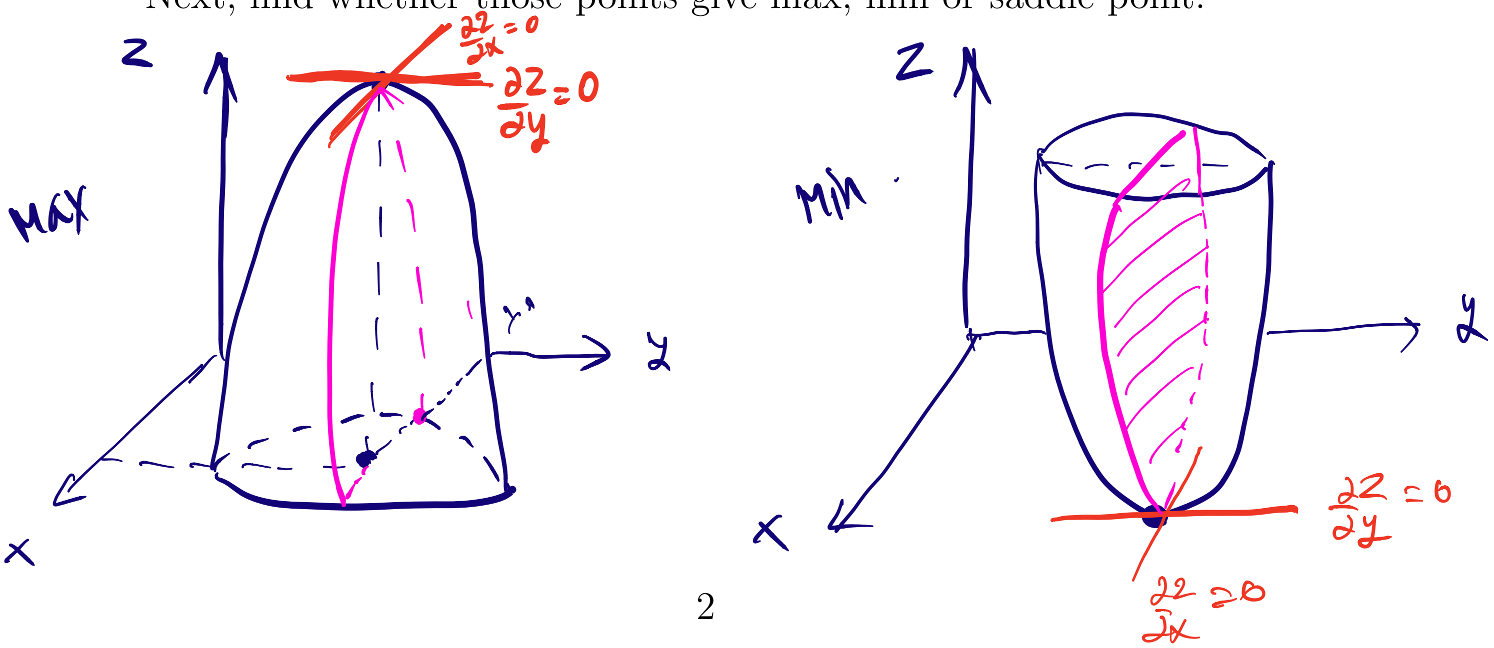
FOC: $\frac{\partial z}{\partial x} = 10 + y - 2x = 0 \Leftrightarrow y = 2x - 10$

$\frac{\partial z}{\partial y} = 10 + x - 2y = 0 \Leftrightarrow 2y = x + 10.$

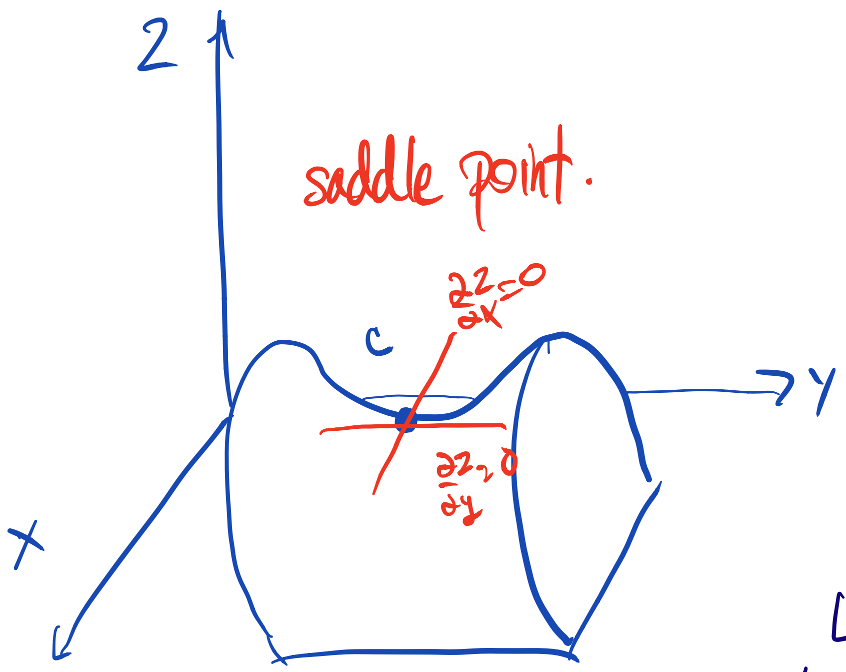
$\Big\} \; x^* = 10, \; y^* = 10.$

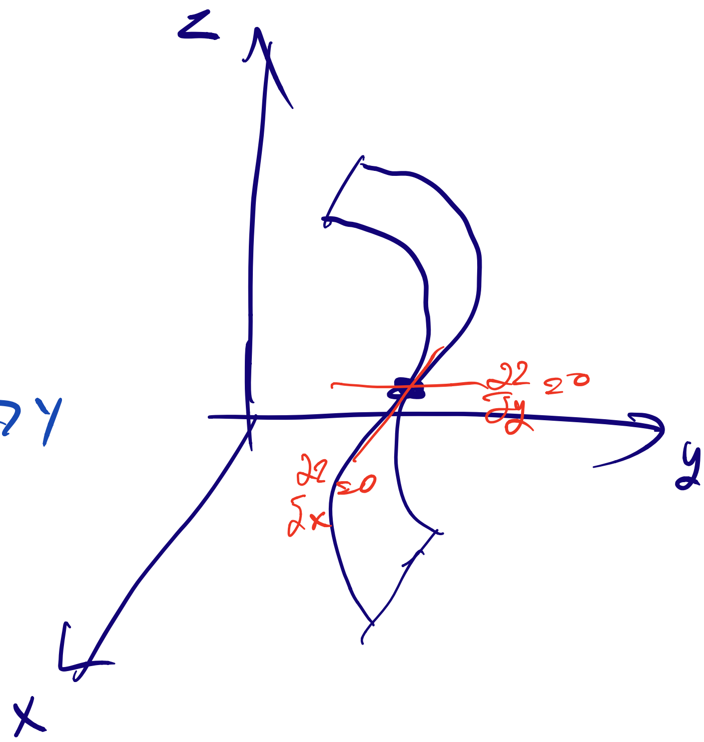
stationary point is $(x^*, y^*) = (10, 10)$
(critical point)

Next, find whether those points give max, min or saddle point.



$\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ are necessary but no yet sufficient for max/min.





3 Second-Order Conditions

$$
\begin{aligned}
d^2z &= d(dz) \qquad \text{suppose } z = f(x,y) \\
&= \frac{\partial}{\partial x} dz dx + \frac{\partial}{\partial y} dz dy \\
&= \frac{\partial}{\partial x}(f_x dx + f_y dy)dx + \frac{\partial}{\partial y}(f_x dx + f_y dy)dy \\
&= (f_{xx}dx + f_{xy}dy)dx + (f_{yx}dx + f_{yy}dy)dy \\
&= f_{xx}dx^2 + f_{xy}dydx + f_{yx}dxdy + f_{yy}dy^2 \\
d^2z &= f_{xx}dx^2 + 2f_{xy}dydx + f_{yy}dy^2 \\
&= \begin{bmatrix} dx & dy \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}}_{\text{Hessian matrix or principal minor}} \begin{bmatrix} dx \\ dy \end{bmatrix}
\end{aligned}
$$

one-variable case:
$f''<0$ max
$f''>0$ min

$$d^2z = f_{xx}dx^2 + 2f_{xy}\,dy\,dx + f_{yy}dy^2 \quad {>0 \atop <0}$$

Note
$$
\begin{aligned}
q &= au^2 + 2huv + bv^2 \\
&= au^2 + 2huv + \tfrac{h^2}{a}v^2 + bv^2 - \tfrac{h^2}{a}v^2 \\
&= a(u^2 + \tfrac{2h}{a}uv + \tfrac{h^2}{a^2}v^2) - (b - \tfrac{h^2}{a})v^2 \\
q &= a(u + \tfrac{h}{a}v)^2 + (\tfrac{ab-h^2}{a})v^2
\end{aligned}
$$

$$d^2z = f_{xx}\left(dx + \frac{f_{xy}}{f_{xx}}dy\right) + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right)dy^2$$

$$= \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$q>0$ when $a>0$ and $ab - h^2>0$

$q<0$ when $a<0$ and $ab - h^2>0$

$$
\begin{aligned}
q &= d^2z \quad , \quad u = dx \\
a &= f_{xx} \quad , \quad v = dy \\
b &= f_{yy} \quad , \quad h = f_{xy}
\end{aligned}
$$

$|H_1| = f_{xx}$

$|H_2| = f_{xx} \cdot f_{yy} - f_{xy}^2$

First leading principal minor $|H_1| = f_{xx}$

Second leading principal minor $|H_2| = f_{xx}f_{yy} - f_{xy}^2$

Second-order sufficient condition

Given that FOC is satisfied

1. maximum when $d^2z<0$ iff $\boxed{f_{xx}<0}$

$f_{xx}<0$ and $f_{xx}f_{yy} > f_{xy}^2$
(−) (−) (+)

$f_{yy}<0$

$\boxed{f_{xx}f_{yy}>(f_{xy})^2}$ or $|H_2|>0$

2. minimum when $d^2z>0$ iff $\boxed{f_{xx}>0}$

$f_{xx}>0$ and $f_{xx}f_{yy} > f_{xy}^2$
(+) (+) (+)

$f_{yy}>0$

$\boxed{f_{xx}f_{yy}>(f_{xy})^2}$ or $|H_2|>0$

Optimization.

one-variable case

$y = f(x)$

$\max_{x} f(x)$

FOC: $f'(x) = 0 \Rightarrow x^*$

SOC: $f''(x) < 0$ max

$f''(x) > 0$ min.

$f''(x) = 0$ inflection point.

2-variable case.

$y = f(x_1, x_2)$

$\max_{x_1, x_2} f(x_1, x_2)$

FOC: $\left.\begin{array}{l} f_1 = \frac{\partial f}{\partial x_1} = 0 \\ f_2 = \frac{\partial f}{\partial x_2} = 0 \end{array}\right\} x_1^*, x_2^*$

SOC: $f_{11} < 0$ <u>and</u> $f_{11} f_{22} - f_{12}^2 > 0$

$\Rightarrow$ max.

$f_{11} > 0$ and $f_{11} f_{22} - f_{12}^2 > 0$

$\Rightarrow$ min

3. otherwise, saddle point

ex. $f(x,y) = 10x + 10y + xy - x^2 - y^2$

$$\text{FOC:} \quad \left.\begin{array}{rcl} f_x & = & 10 + y - 2x \\ f_y & = & 10 + x - 2y \end{array}\right\} \quad x^* = y^* = 10.$$

$$\text{SOC:} \quad \begin{array}{rcl} f_{xx} & = & -2 \\ f_{xy} & = & 1 \\ f_{yx} & = & 1 \\ f_{yy} & = & -2 \end{array}$$

$$\text{H} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$|H_1| = |f_{xx}| = -2 < 0$$

$$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$$

$(10,10) \Rightarrow$ Max

ex. $f(x,y) = -2x^2 - 2xy - 2y^2 + 32x + 42y - 158$

$$\begin{array}{rcl} x^* & = & 5 \\ y^* & = & 8 \end{array}$$

$$\left.\begin{array}{rcl} |H_1| & < & 0 \\ |H_2| & = & 12 \quad > 0 \end{array}\right\} \text{max}$$

<u>ex.</u> $f(x, y) = -x^2 + xy - y^2 + 3x$

$$\begin{aligned} (2,1) \quad z^* &= 3 \\ |H_1| &< 0 \\ |H_2| &> 0 \end{aligned}$$

4 Concavity and Convexity: 2-independent-variable-case

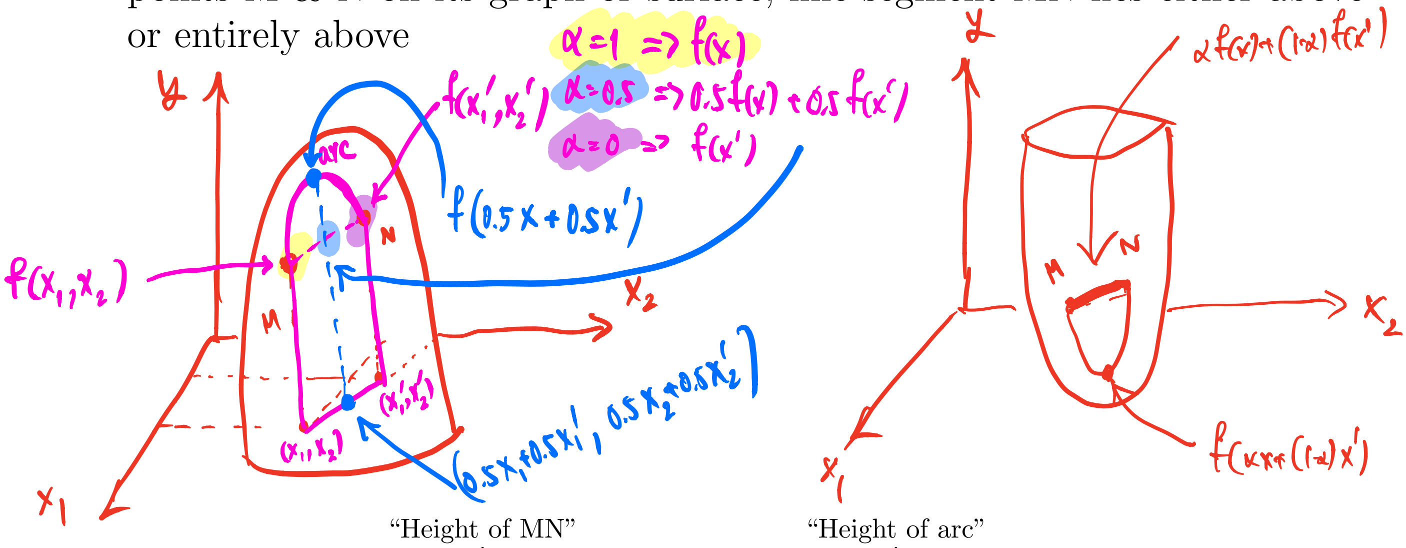
Consider a function with 2 choice variable: $f(x_1, x_2)$

⇒ global max

<u>Definition</u> the function is (strictly) concave iff for any pair of distinct points M & N on its graph or surface, line segment MN lies either below or entirely below

⇒ global min

<u>Definition</u> the function is (strictly) convex iff for any pair of distinct points M & N on its graph or surface, line segment MN lies either above or entirely above



$$\text{or} \qquad \overbrace{\alpha f(x) + (1-\alpha) f(x')}^{\text{“Height of MN”}} < \overbrace{f(\alpha x + (1-\alpha)x')}^{\text{“Height of arc”}} \Rightarrow \text{strictly concave}$$

for $\forall \alpha \in [0,1]$

$$\alpha f(x) + (1-\alpha) f(x') > f(\alpha x + (1-\alpha)x') \Rightarrow \text{strictly convex}$$

If we use derivative conditions;

Consider

$z = f(x_1, x_2)$ - twice continuously differentiable $\rightarrow d^2z$ is defined.

$z = f(x_1, x_2)$ is strictly concave iff d^2z is everywhere definite negative or $\boxed{d^2z<0}$ $f_{11} < 0$ and $f_{11}f_{22} - f_{12}^2 > 0$ $\forall x_1, \forall x_2$

$\Rightarrow z^*$ is max.

$z = f(x_1, x_2)$ is strictly convex iff d^2z is everywhere definite positive or $\boxed{d^2z>0}$ $f_{11} > 0$ and $f_{11}f_{22} - f_{12}^2 > 0$ $\forall x_1, \forall x_2$

$\Rightarrow z^*$ is min.

ex. $z = x_1^2 - x_2^2$

$f_1 = 2x_1 = 0$, $f_2 = -2x_2 = 0$ } $x_1^* = 0$, $x_2^* = 0$

$f_{11} = 2$, $f_{22} = -2$*, $f_{12} = 0$ } $f_{11} = 2 > 0$ ✓

$f_{11}f_{22} - f_{12}^2 = (2)(-2) - 0 = -4 < 0$ X

$\forall x_1, \forall x_2$ ~~strictly convex $\Rightarrow$ global min.~~

ex. $z = x_1^2 + x_2^2$

$f_1 = 2x_1 = 0$, $f_2 = 2x_2 = 0$ } $x_1^* = 0$, $x_2^* = 0$

$f_{11} = 2$, $f_{22} = 2$, $f_{12} = 0$ } $f_{11} = 2 > 0$ ✓

$f_{11}f_{22} - f_{12}^2 = 2(2) - 0 = 4 > 0$ ✓

$\forall x_1, \forall x_2$ strictly convex $\Rightarrow$ global min. at $(0,0)$

ex. $z = 2x^2 - xy + y^2$ (convex)

Application: Duopoly

There are 2 firms with identical cost $TC_i = cQ_i$, where $i = 1, 2$ and market demand $P = a - bQ$, $Q = Q_1 + Q_2$ Find Q_1, Q_2 that maximizes each firm's profit.

$$\text{Firm 1: } \Pi_1 = P(Q) \cdot Q_1 - cQ_1$$

$$= [a - b(Q_1 + Q_2)]Q_1 - cQ_1$$

$$\Pi_1 = (a-c)Q_1 - bQ_1^2 - bQ_1Q_2$$

$$\max_{Q_1} \Pi_1 = (a-c)Q_1 - bQ_1^2 - bQ_1Q_2$$

$$\underline{\text{FOC}} \quad \frac{\partial \Pi}{\partial Q_1} = (a-c) - 2bQ_1 - bQ_2 = 0$$

$$\Rightarrow Q_1^* = \frac{(a-c) - bQ_2}{2b}$$ firm's best response function

$$Q_1^* = \frac{(a-c) - b\left(\frac{(a-c) - bQ_1}{2b}\right)}{2b}$$

$$\text{Firm 2: } \Pi_2 = P(Q) \cdot Q_2 - cQ_2$$

$$\underline{\text{FOC}} \quad Q_2^* = \frac{(a-c) - bQ_1}{2b}$$ firm's best response function

$$= \frac{(a-c) - \frac{(a-c) - bQ_1}{2}}{2b}$$

$$= \frac{2a - 2c - a + c + bQ_1}{4b}$$

$$Q_1^* = \frac{a-c}{3b}$$

$$Q_2^* = \frac{a-c}{3b}$$

$$\Pi_1 = (a-c)Q_1 - bQ_1^2 - bQ_1Q_2$$

$\frac{\partial \pi_1}{\partial Q_1} = (a-c) - 2bQ_1 - bQ_2$

$\frac{\partial^2 \pi_1}{\partial Q_1^2} = -2b$ ✓ $\frac{\partial^2 \pi_1}{\partial Q_1 \partial Q_2} = -b$ ✓

$Q_1 = \frac{a-c+bQ_1}{4b}$

$4bQ_1 = a-c+bQ_1$

$Q_1^* = \frac{a-c}{3b}$

$\frac{\partial \pi_1}{\partial Q_2} = -bQ_1$

$\frac{\partial^2 \pi_1}{\partial Q_2^2} = 0$ ✓

SOC : π_1

f_{11} $\frac{\partial^2 \pi_1}{\partial Q_1^2} = -2b$

f_{22} $\frac{\partial^2 \pi_1}{\partial Q_2^2} = 0$

f_{12} $\frac{\partial^2 \pi_1}{\partial Q_1 \partial Q_2} = -b$

f_{11} $\frac{\partial^2 \pi}{\partial Q_1^2} < 0$

$\frac{\partial^2 \pi}{\partial Q_1^2}\frac{\partial^2 \pi}{\partial Q_2^2} - \left(\frac{\partial^2 \pi}{\partial Q_2 \partial Q_1}\right)^2 = (-2b)(0) - (-b)^2 = -b^2 < 0.$

$f_{11} \cdot f_{22} - f_{12}^2$

ex. Given the following information, market demand $P = 150 - Q$, $Q = Q_1 + Q_2$, $P = P_1 = P_2$ and $TC_1 = 60Q_1$, $TC_2 = 60Q_2$. Find optimal quantity that each firm should produce. Also check for their SOC.

$\pi_1 = (150 - Q_1 - Q_2)Q_1 - 60Q_1$

$= 150Q_1 - Q_1^2 - Q_1Q_2 - 60Q_1$

$= -Q_1^2 - Q_1Q_2 + 90Q_1$

$\frac{\partial \pi_1}{\partial Q_1} = -2Q_1 - Q_2 + 90 = 0.$

$Q_1^* = \frac{Q_2 - 90}{-2}$

$= \frac{90 - Q_2}{2}$

$\pi_2 = (150 - Q_1 - Q_2)Q_2 - 60Q_2$

$\vdots$

$Q_2^* = \frac{90 - Q_1}{2}$

$Q_1^* = 30$

$Q_2^* = 30$

SOC $\frac{\partial^2 \pi_1}{\partial Q_1^2} = -2$ ✓ $< 0.$

$\frac{\partial^2 \pi_1}{\partial Q_2^2} = 0$

$\frac{\partial^2 \pi_1}{\partial Q_1 \partial Q_2} = -1$

$f_{11}f_{22} - f_{12}^2 = (-2)(0) - (-1)^2 = -1 < 0$

Application: Duopoly-extension

1. Cournot: Move at the same time

2. Stackelberg: One seller can move first

Suppose that there are two firms and they face the market demand $P = 30 - Q$, where $Q = q_1 + q_2$. Their marginal cost structures are the same: $MC_1 = MC_2 = 3$. Consider the following scenarios:

1) Both move simultaneously
2) Firm 1 leads/moves first
3) Firm 2 leads/moves first
4) Both try to lead

max $f(x_1, x_2)$

FOC : $f_1 = 0$, $f_2 = 0$ } x_1^*, x_2^*

SOC : $f_{11} < 0$, $f_{11}f_{22} - f_{12}^2 > 0$ } max

$f_{11} > 0$, $f_{11}f_{22} - f_{12}^2 > 0$ } min

otherwise, saddle point.

$$P = 30 - q_1 - q_2$$
$$MC_1 = MC_2 = 3.$$

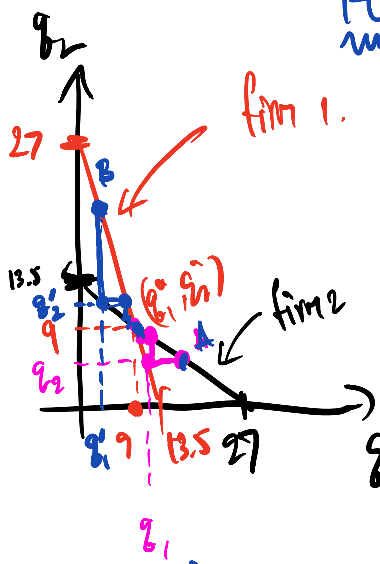
(1) Cournot: □ $\pi_1 = \underbrace{P \cdot q_1}_{\text{revenue}} - \underbrace{C_1}_{\text{cost}}$ $\quad TC = MC_1 \cdot q_1$ $\quad \frac{\Delta TC}{\Delta q_1}$

$$= (30 - q_1 - q_2) \cdot q_1 - 3q_1$$
$$= 30q_1 - q_1^2 - q_1 q_2 - 3q_1$$

$0.5q_2 = 13.5 - q_1$

$q_2 = 27 - 2q_1$

② Draw reaction fn's



$q_1^* = 9,\ q_2^* = 9$

FOC $\frac{\partial \pi_1}{\partial q_1} = 30 - 2q_1 - q_2 - 3 = 0.$

✓ $\boxed{q_1 = 13.5 - 0.5q_2}$

□ $\pi_2 = P \cdot q_2 - C_2$

$$= (30 - q_1 - q_2) \cdot q_2 - 3q_2$$

FOC $\frac{\partial \pi}{\partial q_2} \Rightarrow$ ✓ $\boxed{q_2 = 13.5 - 0.5q_1}$

reaction functions

① $\left.\begin{array}{l} q_1 = 13.5 - 0.5q_2 \\ q_2 = 13.5 - 0.5q_1 \end{array}\right\}$ directly sol.

$q_1^* = 9,\ q_2^* = 9.$

$\pi_1 = 30q_1 - q_1^2 - q_1 q_2 - 3q_1$

$\pi_1 = 76$

$\pi_2 = 76$

SOC.

★ (2) Firm 1 moves first.

$$\pi_1 = (30 - q_1 - q_2) \cdot q_1 - 3q_1$$

Assume perfect knowledge ⇒ ① knows reaction fn of ② and take ②'s behavior into a/c.

$$\pi_1 = \left[30 - q_1 - (13.5 - 0.5q_1)\right] q_1 - 3q_1$$
$$= 30q_1 - q_1^2 - 13.5q_1 + 0.5q_1^2 - 3q_1$$

FOC $\frac{\partial \pi}{\partial q_1} = 30 - 2q_1 - 13.5 + q_1 - 3 = 0.$

$q_1^* = 13.5 \Rightarrow q_2^* = 13.5 - 0.5(13.5) = \underline{6.75}$ (2's reaction function)

$\pi_1^* = 91.125$

$\pi_2^* = 45.5625$

$Q = q_1 + q_2 = 13.5 + 6.75$

(3) Firm 2 moves first.

$\boxed{q_1 = 13.5 - 0.5 q_2}$ 1's reaction function

$\pi_2 = P \cdot q_2 - C_2 = (30 - q_1 - q_2) q_2 - 3 q_2$

$= (30 - (13.5 + 0.5 q_2) - q_2) q_2 - 3 q_2$

$\vdots$ FOC $\frac{\partial \pi_2}{\partial q_2} = 0$

$q_2^* = 13.5 \Rightarrow q_1 = 13.5 - 0.5(13.5) = 6.75$

$\pi_2^* = 86.125 \Rightarrow 91.125$

$\pi_1^* = 58.5625 \Rightarrow 45.5625$

(4) Both firms try to lead.

① $\pi_1 = (30 - q_1 - (13.5 - 0.5 q_1)) q_1 - 3 q_1 \xrightarrow{FOC} q_1^* = 13.5$

$\pi_2 = (30 - (13.5 - 0.5 q_2) - q_2) q_2 - 3 q_2 \xrightarrow{FOC} q_2^* = 13.5$

PC.
✓ monopoly.
✓ monopolistic c~
Oligoly

$\Rightarrow \pi_1^* = -5 = 0$

$\pi_2^* = -5 = 0$

Firm 1st → lead $\Rightarrow q_1^*, q_2^*$

2nd → follow $\Rightarrow q_1'^*, q_2'^*$

"Game theory." Nash

(π_1, π_2)	lead	follow
lead	(0, 0)	* (86.125, 58.5625)
follow	* (58.5625, 86.125)	(76, 76)

"Nash equilibrium" EE311.

Application: Multiproduct firm

1. Suppose a firm produce two goods where both of them are selling into perfectly competitive markets. Given all the following information of market structure: $P_1 = 6$, $P_2 = 9$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$. Find Q_1 and Q_2 that maximize its profit Π

$$\Pi(Q_1, Q_2) = TR_1 + TR_2 - TC$$

$$\max_{Q_1,Q_2} \Pi(Q_1, Q_2) = 6Q_1 + 9Q_2 - (2Q_1^2 + Q_1Q_2 + 2Q_2^2)$$

$\boxed{PC.}$ Q_1^*, Q_2^* : FOC. $\frac{\partial \Pi}{\partial Q_1} = 6 - 4Q_1 - Q_2 = 0$

$\frac{\partial \Pi}{\partial Q_2} = 9 - Q_1 - 4Q_2 = 0.$

$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

$\Downarrow$

$Q_1^* = 1$, $Q_2^* = 2$.

SOC. $\Pi_{11} = -4$

$\Pi_{22} = -4$

$\Pi_{12} = -1$

$\Pi_{11} < 0$

$\Pi_{11}\Pi_{22} - \Pi_{12}^2 = \underbrace{(-4)(-4)-(-1)^2}_{16-1=15} > 0$

$\boxed{(1,2) \text{ yield max. profit}}$

Suppose instead that this particular firm operate as a monopoly for both goods. Consider all the following information: $P_1 = 35 - Q_1$, $P_2 = 33 - Q_2$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$. Find Q_1 and Q_2 that maximize Π

$$\Pi(Q_1, Q_2) = TR_1 + TR_2 - TC$$

$$\max_{Q_1,Q_2} \Pi(Q_1, Q_2) = (35 - Q_1)Q_1 + (33 - Q_2)Q_2 - (2Q_1^2 + Q_1Q_2 + 2Q_2^2)$$

$\boxed{M}$ Q_1^*, Q_2^* : FOC $\frac{\partial \Pi}{\partial Q_1} = 35 - 2Q_1 - 4Q_1 - Q_2 = 0.$ $6Q_1 + Q_2 = 35$

$\frac{\partial \Pi}{\partial Q_2} = 33 - 2Q_2 - Q_1 - 4Q_2 = 0$ $Q_1 + 6Q_2 = 33$

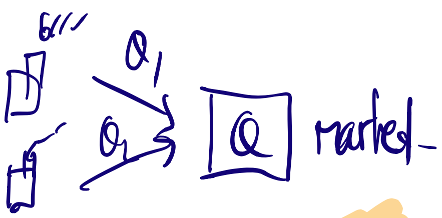
$35Q_1 - Q_1^2 - 33Q_2 + Q_2^2 - 2Q_1^2 - Q_1Q_2 - 2Q_2^2$

$-3Q_1^2 - Q_2^2 - 35Q_1 - 33Q_2$

$$\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \end{bmatrix} \Rightarrow \left.\begin{matrix} Q_1^* = 5.05 \\ Q_2^* = 4.66 \end{matrix}\right\} \text{max profit}$$

SOC $\left.\begin{matrix} \pi_{11} = -6 \\ \pi_{22} = -6 \\ \pi_{12} = -1 \end{matrix}\right\} \begin{matrix} \pi_{11} < 0. \\ \pi_{11}\pi_{22} - \pi_{12}^2 > 0 \end{matrix}$

multiproduct: $q_1 \to P_1$, $q_2 \to P_2$



Application: Multiplant firm

Consider a firm that operate in perfectly competitive market with $P = 25$ and has two factories which each has the following cost structures $TC_1 = 2Q_1^2 + 5Q_1 + 10$ and $TC_2 = 2Q_2^2 + 3Q_2 + 15$. Find Q_1 and Q_2 that maximize firm's profit: $\Pi(Q_1, Q_2) = TR_1 - TC_1 - TC_2$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2) = \boxed{25}(Q_1 + Q_2) - (2Q_1^2 + 5Q_1 + 10) - (2Q_2^2 + 3Q_2 + 15)$$

FOC. $\left.\begin{matrix} \dfrac{\partial \pi}{\partial Q_1} = 20 - 4Q_1 = 0. \Rightarrow Q_1^* = 5 \\ \dfrac{\partial \pi}{\partial Q_2} = 22 - 4Q_2 = 0. \Rightarrow Q_2^* = 5.5 \end{matrix}\right\}$ max.

SOC $H = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \Rightarrow \begin{matrix} \pi_{11} = -4 \\ \pi_{11}\pi_{22} - \pi_{12}^2 > 0. \end{matrix}$

Application: Multimarket Monopoly or Price discrimination

Suppose that a firm has certain market power over 2 goods. The total revenue for goods 1 and 2 is $R = R_1(Q_1) + R_2(Q_2)$ and $C = C(Q)$ where $Q = Q_1 + Q_2$. Find FONC and SOSC for maximum profit Π

$$\Pi = R_1(Q_1) + R_2(Q_2) - C(Q) = R_1(Q_1) + R_2(Q_2) - C(Q_1 + Q_2)$$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2)$$

$Q = Q_1 + Q_2$, $\frac{\partial Q}{\partial Q_1} = 1$

FONC

$$\frac{\partial \Pi}{\partial Q_1} = R_1' - C' \frac{\partial Q}{\partial Q_1} = 0 \Rightarrow R_1' = C' \Leftrightarrow MR_1 = MC$$

$$\frac{\partial \Pi}{\partial Q_2} = R_2' - C' \frac{\partial Q}{\partial Q_2} = 0 \Rightarrow R_2' = C' \Leftrightarrow MR_2 = MC$$

$\Big\}$ $MR_1 = MR_2 = MC$.

Solve for (Q_1^*, Q_2^*): Firms need to produce until: $\boxed{MR_1 = MR_2 = MC}$

$\Downarrow$ Q_1^*, Q_2^*

SOSC

$$\Pi_{11} = R_1'' - C''$$

$$\Pi_{22} = R_2'' - C''$$

$$\Pi_{12} = \Pi_{21} = \frac{\partial}{\partial Q_1}\left[R_2' - C'(Q)\right] = -C''(Q)$$

SOSC is satisfied when 1. $\overbrace{R_1'' - C''}^{\Pi_{11}} < 0$ ✓

2. $\overbrace{R_2'' - C''}^{\Pi_{22}} < 0$

3. $\Pi_{11}\Pi_{22} - \Pi_{12}^2 > 0$ ✓

$\Rightarrow Q_1^*, Q_2^*$ give maximum profit.

ex. $P_1 = 22 - 2Q_1, P_2 = 10 - 0.5Q_2$ and$TC = 2Q + 5$. Find Q_1, Q_2, P_1 and P_2 that maximize firm's profit.

$$\max_{Q_1,Q_2} \Pi(Q_1, Q_2) = (22 - 2Q_1)Q_1 + (10 - 0.5Q_2) - (2Q + 5)$$

$Q_1^* = 5,\ Q_2^* = 8,$ SOC ✓ $\boxed{\Pi_{11},\ \Pi_{11}\Pi_{22} - \Pi_{12}^2}$

Application: Input decision of a firm

Let $Q = f(K, L) = 5K^{0.5}L^{0.25},\ P = 4,\ w = 10,\ r = 5$. Find $K^*,\ L^*$ max profit.

$$\Pi = P \cdot Q - \underbrace{TC}_{\text{Cost}}$$

Revenue Cost

$TC = r \cdot K + w \cdot L$

$$= 4 \cdot (5K^{0.5}L^{0.25}) - (10K + 5L)$$

$$\max_{K,L} \Pi(K, L)$$

$$\text{FONC}: \quad \frac{\partial \Pi}{\partial K} = 10K^{-0.5}L^{0.25} - 5 = 0 \quad \Rightarrow \quad 2K^{-0.5}L^{0.25} - 1 = 0$$

$$\frac{\partial \Pi}{\partial L} = 5K^{-0.5}L^{-0.75} - 10 = 0 \quad \Rightarrow \quad K^{0.5}L^{-0.75} - 2 = 0$$

$$\boxed{\begin{aligned} K^* &= 4 \\ L^* &= 1 \end{aligned}}$$

max $f(x_1, x_2)$

FOC: $f_1 = 0$ $\Rightarrow$ SOC

$f_2 = 0$.

$$\underline{SOSC} \quad \Pi_{KK} = -5K^{-1.5}L^{0.25} \Rightarrow \Pi_{KK}(K^*=4, L^*=1) = -\tfrac{5}{8}<0$$

$$\Pi_{LL} = -3.75K^{0.5}L^{-1.75} \Rightarrow \Pi_{LL}(K^*=4, L^*=1) = -\tfrac{15}{2}<0$$

$$\Pi_{KL} = 2.5K^{-0.5}L^{-0.75} \Rightarrow \Pi_{KL}(K^*=4, L^*=1) = -\tfrac{5}{4}$$

$$H = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} \\ \frac{5}{4} & -\frac{15}{2} \end{bmatrix} \qquad |H_1|<0\ ,\ |H_2|>0 \therefore \Pi(4,1) \text{ is max.}$$

5 Comparative-Static Aspects of Optimization

We can use partial differentiation to study how a change in exogenous variables affect the equilibrium outcome in the model.

ex. Consider a perfectly competitive markets. P_1 and P_2 are exogenous variables. The total revenue of the firm is $R = P_1Q_1 + P_2Q_2$, and its cost is $C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$. Find how equilibrium quantity of goods 1 and 2 will be affected after a change in market prices P_1 and P_2.

$$\max_{Q_1,Q_2} \Pi(Q_1,Q_2) = P_1Q_1 + P_2Q_2 - (2Q_1^2 + Q_1Q_2 + 2Q_2^2)$$

$$\boxed{\frac{\partial Q_1^*}{\partial P_1}, \frac{\partial Q_1^*}{\partial P_2} \quad \frac{\partial Q_2^*}{\partial P_1}, \frac{\partial Q_2^*}{\partial P_2}}$$

Q_1^*, Q_2^* : FOC $\quad Q_1^* = \dfrac{4P_1 - P_2}{15}$ ✓

$Q_2^* = \dfrac{4P_2 - P_1}{15}$ ✓

$$\frac{\partial Q_1^*}{\partial P_1} = \frac{4}{15} \qquad \frac{\partial Q_2^*}{\partial P_1} = \frac{-4}{15}$$

$$\frac{\partial Q_1^*}{\partial P_2} = \frac{-1}{15} \qquad \frac{\partial Q_2^*}{\partial P_2} = \frac{1}{15}.$$

ex. Consider $R = P \cdot Q(K, L)$ where P, w, and r are exogenous. Find $\frac{\partial L^*}{\partial P}, \frac{\partial K^*}{\partial P}$. Given, $C = wL + rK$

Revenue Cost. MB MC.

$$\max_{K,L} \Pi(K, L) = P \cdot Q(K, L) - (wL + rK)$$

FONC $\Pi_K = P \cdot Q_K - r = 0 \Rightarrow P.Q_K = r, \Rightarrow K^*$

$\Pi_L = P \cdot Q_L - w = 0 \Rightarrow P.Q_L = w; \Rightarrow L^*$

$$\Rightarrow K^* = K^*(P, w, r)\ ,\ L^* = L^*(P, w, r)$$

If P changes, while w and r remain constant; what will happen to K^*, L^*?

$$K^* = K^*(P) = K^*(P, \bar{w}, \bar{r})$$

$$L^* = L^*(P) = L^*(P, \bar{w}, \bar{r})$$

For FONC:

Total derivative.

$$\Pi_K = P \cdot Q_K(K^*(P), L^*(P)) - r = 0 \Rightarrow K^*$$

$$\Pi_L = P \cdot Q_L(K^*(P), L^*(P)) - w = 0$$

$$\boxed{\frac{d\Pi_K}{dP}} = \frac{d}{dP}\left[P.Q_K(K^*(P), L^*(P)) - r\right]$$

$$= P.\left[\frac{dQ_K}{dK}\frac{dK}{dP} + \frac{dQ_K}{dL}\cdot\frac{dL}{dP}\right] - \frac{dr}{dP} \nearrow 0$$

$$\frac{d\Pi_K}{dP} = P\, Q_{KK}\frac{dK^*}{dP} + P Q_{KL}\frac{dL^*}{dP} + Q_K$$

$$\therefore \cancel{\frac{d\Pi_K}{dP}} =$$

$$\frac{d\Pi_L}{dP} = P \cdot Q_{KL}\frac{dK^*}{dP} + P \cdot Q_{LL}\frac{dL^*}{dP} + Q_L$$

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$$\begin{bmatrix} P \cdot Q_{KK} & P \cdot Q_{LK} \\ P \cdot Q_{KL} & P \cdot Q_{LL} \end{bmatrix} \begin{bmatrix} \frac{dK^*}{dP} \\ \frac{dL^*}{dP} \end{bmatrix} = \begin{bmatrix} -Q_K \\ -Q_L \end{bmatrix}$$

Total differential.

$d\Pi_K = \cancel{P.Q_{KK}dK + PQ_{KL}dL} - 0.$

$\frac{d\Pi_K}{dP} = \cancel{P\,Q_{KK}\frac{dK}{dP} + PQ_{KL}\frac{dL}{dP}}$

$P.\frac{dQ_K}{dP} + Q_K\frac{dP}{dP} - r$

$P.\left[\frac{dQ_K}{dK}\frac{dK}{dL} + \frac{dQ_K}{dL}\frac{dP}{dL}\right] + Q_K$

Check the answers & signs.

$\frac{dK^*}{dP} = \frac{-P\cdot Q_K Q_{LL} + P\cdot Q_L Q_{LK}}{P^2(Q_{KK}Q_{LL} - Q_{LK}^2)} > 0$ if $Q_{LK} > 0$

$\frac{dL^*}{dP} = (-PQ_L Q_{KK} + PQ_K Q_{LK}) / [P^2(Q_{KK}Q_{LL} - Q_{LK}^2)] > 0$ if $Q_{LK} > 0$

$\therefore$ L and K are complement

6 Multivariable Optimization

max $f(x_1, x_2)$

FOC: $f_1 = 0, f_2 = 0$

SOC: max $f_{11} < 0$

$f_{11}f_{22} - f_{12}^2 > 0$

min $f_{11} > 0$

$f_{11}f_{22} - f_{12}^2 > 0$

Let a function with 3 choice variables:

$$\begin{aligned}
z &= f(\overset{K}{x_1}, \overset{L}{x_2}, \overset{e}{x_3}) \\
dz &= f_1 dx_1 + f_2 dx_2 + f_3 dx_3
\end{aligned}$$

✓ FONC: $dz = 0 \Leftrightarrow f_1 = f_2 = f_3 = 0 \Rightarrow x_1^*, x_2^*, x_3^*$

✓ SOSC:

$$\begin{aligned}
d^2z &= \frac{\partial}{\partial x_1}(dz)dx_1 + \frac{\partial}{\partial x_2}(dz)dx_2 + \frac{\partial}{\partial x_3}(dz)dx_3 \\
&= \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix} \underbrace{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}}_{\text{Hessian Matrix}} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}
\end{aligned}$$

Given any Hessian matrix: $\boxed{H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}}$

$|H_1| = |f_{11}|$ first leading principle minor.

$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$ second ______

$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$ third ______

$$f_{11}f_{22}-f_{12}^2$$

$$\left.\begin{array}{rcl} |H_1| & = & f_{11} \\ |H_2| & = & \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \\ |H_3| & = & \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \end{array}\right\}$$

x_1^*, x_2^*, x_3^* ↓ $d^2z<0$ (max)	x_1^*, x_2^*, x_3^* ↓ $d^2z>0$ (min)
$\lvert H_1\rvert<0$ (−) ✓	→ $\lvert H_1\rvert>0$ (+)
$\lvert H_2\rvert>0$ (+)	→ $\lvert H_2\rvert>0$ (+)
$\lvert H_3\rvert<0$ (−)	→ $\lvert H_3\rvert>0$ (+)
"negative definite"	"positive definite"
⇒ all principal minors alternate its sign	⇒ all principal minor must be positive

Summary $f(x_1, x_2, x_3, \dots, x_n)$

	max	min
FONC	$f_1=f_2=\cdots=f_n=0$	$f_1=f_2=\cdots=f_n=0$
SOSC	$\lvert H_1\rvert<0$, $\lvert H_2\rvert>0$ $\lvert H_3\rvert<0, \dots \lvert H_4\rvert>0$ or $\boxed{(-1)^i\lvert H_i\rvert>0}$ for $i=1,2,\dots,n$	$\lvert H_1\rvert>0$, $\lvert H_2\rvert>0$ $, \lvert H_3\rvert>0, \dots$ or $\lvert H_i\rvert>0$ for $i=1,2,\dots,n$

$(-1)^i|H_i|>0$ for $i=1,2,\dots,n$

ex. $f(x_1,x_2,x_3,x_4,x_5)$: max. conditions? // min conditions?

$|H_1|<0$ $\boxed{(-1)^i|H_i|>0}$ $i=1,2,3,4,5$.

$|H_2|>0$ $i=1 \Rightarrow (-1)^1|H_1|>0$

$|H_3|<0$. $|H_1|<0$ ✓

$|H_4|>0$. $i=5 \Rightarrow (-1)^5|H_5|>0$

$|H_5|<0$ × $|H_5|<0$ ✓

otherwise, saddle point.

$|H_1|>0$

$|H_2|>0$

$|H_3|>0$

$|H_4|>0$

$|H_5|>0$

<u>ex.</u> $y = x_1^2 + 6x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3$ $\quad f(x_1, x_2, x_3)$

$$f_1 = \frac{\partial y}{\partial x_1} = 2x_1 - 2x_2 = 0$$

$$f_2 = \frac{\partial y}{\partial x_2} = 12x_2 - 2x_1 - 4x_3 = 0$$

$$f_3 = \frac{\partial y}{\partial x_3} = 6x_3 - 4x_2 = 0$$

x_1^*, x_2^*, x_3^*

$$\left.\begin{array}{lcl} f_{11} & = & 2 \\ f_{22} & = & 12 \\ f_{33} & = & 6 \\ f_{12} & = & -2 \\ f_{13} & = & 0 \\ f_{21} & = & -2 \\ f_{23} & = & -4 \\ f_{31} & = & 0 \\ f_{32} & = & -4 \end{array}\right\} H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 12 & -4 \\ 0 & -4 & 6 \end{bmatrix}$$

$$|H_1| = 2 > 0$$

$$|H_2| = \begin{vmatrix} 2 & -2 \\ -2 & 12 \end{vmatrix} = 20 > 0$$

$$|H_3| = |H| = 88 > 0$$

$(+)$

$\therefore d^2y > 0$ positive definite (min)

Application: Multimarket Monopoly

$$\begin{aligned} \text{Let} \quad P_1 &= 63 - 4Q_1 \\ P_2 &= 105 - 5Q_2 \\ P_3 &= 75 - Q_3 \\ TC &= 15Q + 20 \end{aligned}$$

Find Q_1 and P_1 for $i = 1, 2, 3$ that max Π

$$\max_{Q_1, Q_2, Q_3} \Pi(Q_1, Q_2, Q_3)$$

$$= (63 - 4Q_1)Q_1 + (105 - 5Q_2)Q_2 + (75 - Q_3)Q_3 - [15(Q_1 + Q_2 + Q_3) + 20]$$

$$\underline{FONC} \quad \begin{aligned} \Pi_1 &= 48 - 8Q_1 &\Rightarrow \quad Q_1^* = 6 \\ \Pi_2 &= 90 - 10Q_2 &\Rightarrow \quad Q_2^* = 9 \\ \Pi_3 &= 60 - 20Q_3 &\Rightarrow \quad Q_3^* = 30 \end{aligned}$$

$$\underline{SOSC} \quad \left. \begin{aligned} \Pi_{11} &= -8 \\ \Pi_{22} &= -10 \\ \Pi_{33} &= -2 \\ \Pi_{12} &= \Pi_{13} = 0 \\ \Pi_{21} &= \Pi_{23} = 0 \\ \Pi_{31} &= \Pi_{33} = 0 \end{aligned} \right\} H = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$|H_1| = -8 < 0$ ✓ $-$

$|H_2| = 80 > 0$ ✓ $+$

$|H_3| = -160 < 0$ ✓ $-$

$\therefore d^2 z < 0$ negative definite (max.)

$\boxed{\text{Quiz 3 on Nov 17, 2020.}}$ Ch 8.

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