EE 320

Notes on Constrained Maximization Problems (Output Maximization and Cost Minimization)

I. Output Maximization Problem

$$\max_{K,L} Q = Q(K,L)$$

Subject to $C_0 = wL + rK$

Lagrangian function:

$$\mathcal{L} = Q(K,L) + \lambda[C_0 - wL - rK] .$$

FONC:

$$\mathcal{L}_K = Q_K - \lambda r = 0 \qquad \text{-- (1)}$$

$$\mathcal{L}_L = Q_L - \lambda w = 0 \qquad \text{-- (2)}$$

$$\mathcal{L}_\lambda = C_0 - wL - rK = 0 \qquad \text{-- (3)}$$

Note: Q_K=marginal product of capital, and Q_L=marginal product of labor.

From (1) and (2), we have:

$$\lambda = \frac{Q_K}{r} = \frac{Q_L}{w} \quad \Rightarrow \quad \frac{Q_K}{Q_L} = \frac{r}{w} \qquad \text{-- (*)}$$

Where $\frac{Q_K}{Q_L}$ = marginal rate of technical substitution of K for L, and $\frac{r}{w}$ =ratio of input prices.

Economic interpretation:

1. $\frac{Q_K}{Q_L}$ is the negative of the slope of the isoquant Q_0 = Q(K, L).

For a given Q_0, totally differentiating Q_0 = Q(K,L) gives:

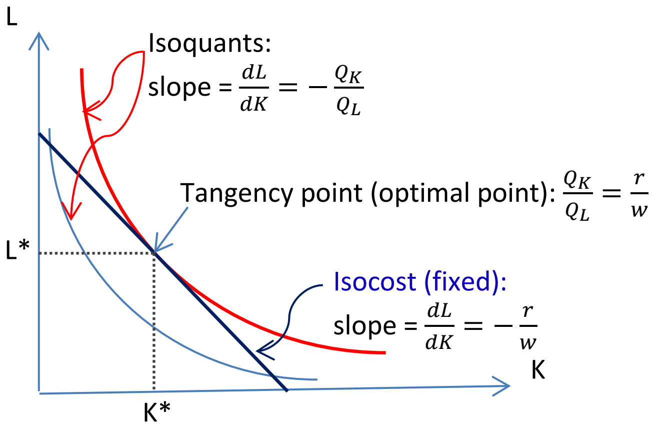
$$dQ_0 = Q_K dK + Q_L dL = 0$$

$\Longrightarrow \frac{dL}{dK} = -\frac{Q_K}{Q_L}$, where $\frac{dL}{dK}$ is the slope of the isoquant.

2. $\frac{r}{w}$ is the negative of the slope of the isocost curve:

From $C_0 = wL + rK$, we can write $L = \frac{C_0}{w} - \frac{r}{w}K$.

Graph



SOSC: For (K*, L*) to give a <u>maximum output</u> level, we need

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{vmatrix} = \begin{vmatrix} 0 & r & w \\ r & Q_{KK} & Q_{KL} \\ w & Q_{LK} & Q_{LL} \end{vmatrix} > 0 .$$

Numerical example:

$$\max_{K,L} Q = KL$$

Subject to $C_0 = 60 = 10K + 6L$

Lagrangian function:

$$\mathcal{L} = KL + \lambda[60 - 10K - 6L] .$$

FONC:

$$\mathcal{L}_K = L - 10\lambda = 0 \qquad \text{-- (1)}$$

$$\mathcal{L}_L = K - 6\lambda = 0 \qquad \text{-- (2)}$$

$$\mathcal{L}_\lambda = 60 - 10K - 6L = 0 \qquad \text{-- (3)}$$

From (1) and (2), we have: $\lambda = \frac{L}{10} = \frac{K}{6}$ ➔ $L = \frac{5}{3}K$ -- (4)

Sub L from (4) in (3), we have: $60 - 10K - 6\left(\frac{5}{3}K\right) = 0$ ➔ $60 - 20K = 0$ ➔ $K^* = 3$.

Sub K* back in (4), we get $L^* = 5$.

SOSC:

$$|\bar{H}| = \begin{vmatrix} 0 & r & w \\ r & Q_{KK} & Q_{KL} \\ w & Q_{LK} & Q_{LL} \end{vmatrix} = \begin{vmatrix} 0 & 10 & 6 \\ 10 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} = -10\begin{vmatrix} 10 & 1 \\ 6 & 0 \end{vmatrix} + 6\begin{vmatrix} 10 & 0 \\ 6 & 1 \end{vmatrix} = 120 > 0 .$$

[Note: $Q_{KK} = 0$; $Q_{LL} = 0$; $Q_{KL} = Q_{LK} = 1$.]

Thus, $(K^*, L^*) = (3, 5)$ is the input combination that gives the maximum output level.

II. Cost Minimization Problem

$$\min_{K,L} C = rK + wL$$

Subject to $\bar{Q} = Q(K, L)$

Lagrangian function:

$$\mathcal{L} = rK + wL + \tilde{\lambda}[\,\bar{Q} - Q(K, L)]\,.$$

FONC:

$$\mathcal{L}_K = r - \tilde{\lambda} Q_K = 0 \qquad \text{-- (1)}$$

$$\mathcal{L}_L = w - \tilde{\lambda} Q_L = 0 \qquad \text{-- (2)}$$

$$\mathcal{L}_\lambda = \bar{Q} - Q(K, L) = 0 \quad \text{-- (3)}$$

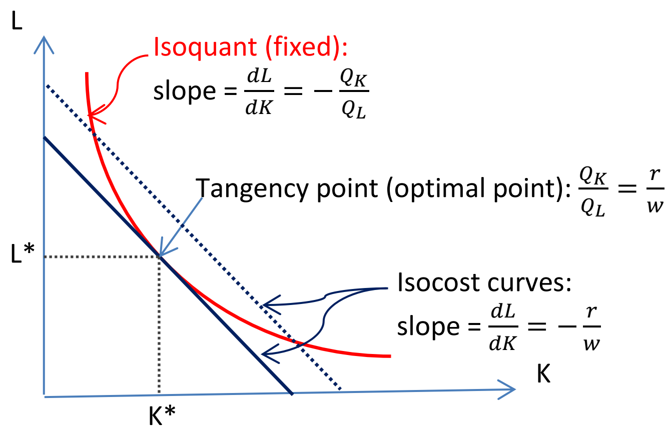
where Q_K=marginal product of capital, and Q_L=marginal product of labor.

From (1) and (2), we have:

$$\tilde{\lambda} = \frac{r}{Q_K} = \frac{w}{Q_L} \quad \Rightarrow \quad \frac{Q_K}{Q_L} = \frac{r}{w} \qquad \text{-- (**)}$$

We can see that the optimal condition (**) for the cost-minimization problem is exactly the same as the optimal condition (*) for the output-maximization problem.

Graph



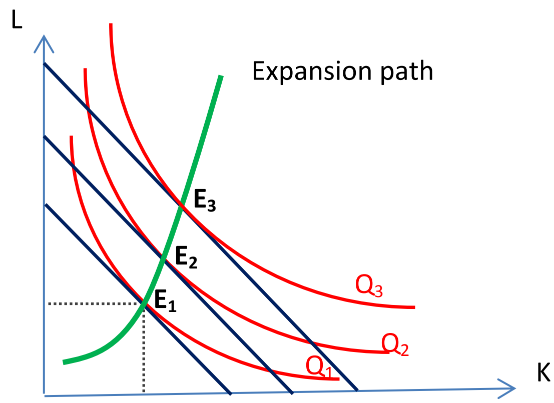
SOSC: For (K*, L*) to be the least-cost input combination, we need

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{vmatrix} = \begin{vmatrix} 0 & Q_K & Q_L \\ Q_K & -\tilde{\lambda} Q_{KK} & -\tilde{\lambda} Q_{KL} \\ Q_L & -\tilde{\lambda} Q_{LK} & -\tilde{\lambda} Q_{LL} \end{vmatrix} < 0\,.$$

Some Comparative Static Analysis of Cost Minimization Problem

Assume a fixed input price ratio. Suppose that the fixed quantity ($\bar{Q}$) increases, what happens to the optimal input ratio $\frac{K^*}{L^*}$?

Consider the following graph.



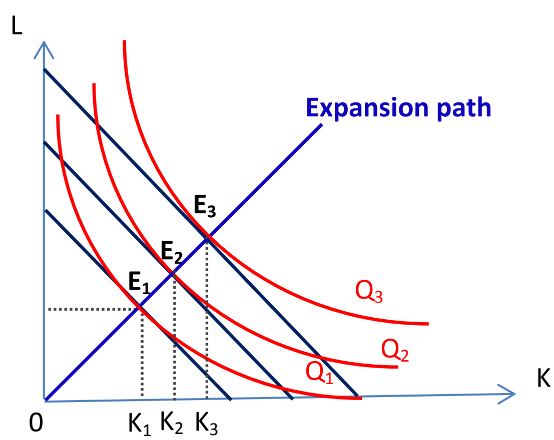
The expansion path is the locus of the tangency points between the isoquants and isocost curves, and it describes the least-cost combinations required to produce varying levels of the fixed quantity.

<u>Special case of expansion path</u>

Consider the Cobb-Douglas production function: $Q = AK^{\alpha}L^{\beta}$.

The expansion path must satisfy:

$$\frac{r}{w} = \frac{Q_K}{Q_L} = \frac{A\alpha K^{\alpha-1}L^{\beta}}{A\beta K^{\alpha}L^{\beta-1}} = \frac{\alpha L}{\beta K} \rightarrow \frac{L^*}{K^*} = \frac{\beta r}{\alpha w} = \text{constant.}$$



In this special case, $\frac{E_1K_1}{OK_1} = \frac{E_2K_2}{OK_2} = \frac{E_3K_3}{OK_3}$.

Numerical example:

$$\min_{K,L} C = 10K + 6L$$

Subject to $\bar{Q} = 15 = KL$

Lagrangian function:

$$\mathcal{L} = 10K + 6L + \tilde{\lambda}[\,15 - KL]\,.$$

FONC:

$$\mathcal{L}_K = 10 - \tilde{\lambda}L = 0 \qquad \text{-- (1)}$$

$$\mathcal{L}_L = 6 - \tilde{\lambda}K = 0 \qquad \text{-- (2)}$$

$$\mathcal{L}_\lambda = 15 - KL = 0 \qquad \text{-- (3)}$$

From (1) and (2), we have:

$$\tilde{\lambda} = \frac{10}{L} = \frac{6}{K} \quad \Rightarrow \quad K = \frac{3}{5}L \qquad \text{--(4)}$$

Sub K in (3), we get: $15 - \left(\frac{3}{5}L\right)L = 0$ ➔ $L^2 = 25$ ➔ $L^* = 5$.

Sub L* back in (4), we have: $K^* = 3$.

SOSC:

At the critical values, $Q_K = L^* = 5$ and $Q_L = K^* = 3$.

$Q_{KK} = 0$; $Q_{LL} = 0$; and $Q_{KL} = Q_{LK} = -1$

$\tilde{\lambda} = \frac{10}{L^*} = 2$.

$$|\bar{H}| = \begin{vmatrix} 0 & Q_K & Q_L \\ Q_K & -\tilde{\lambda}Q_{KK} & -\tilde{\lambda}Q_{KL} \\ Q_L & -\tilde{\lambda}Q_{LK} & -\tilde{\lambda}Q_{LL} \end{vmatrix} = \begin{vmatrix} 0 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & 0 \end{vmatrix} = -60 < 0\,.$$

Thus, the SOSC for a minimum are satisfied. $(K^*, L^*) = (3, 5)$ is the least-cost input combination.