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Question 0:

Consider the function f defined for all (x,y) such that

$$f(x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1-a^3, a^2)$ is a stationary point of $f(x, y; a)$.
b. State the condition under which the above stationary point is a global maximum.
c. Given that $G(a) = f(x^*, y^*; a)$, show the derivative of G with respect to a.
d. Calculate $\frac{\partial f(x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1-a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?
e. Determine the domain of (x,y) in the xy-plan where f is convex.

a) $\frac{df}{dx} = x - 1 + ay = 0 \Rightarrow x = 1 - ay$

$\frac{df}{dy} = ax - a - y^2 + 2a^2y = 0 \Rightarrow y^2 - 2a^2y = ax - a$

sub x in; $y^2 - 2a^2y = a(1-ay) - a$

$y^2 - 2a^2y + a^2y = 0$

$y(y - 2a^2 + a^2) = 0$

$y^* = 0, a^2$

$x^* = 0, 1-a^3$

therefore, the stationary point of $f(x,y;a)$ from the solution is $(x^*, y^*) = (1-a^3, a^2)$

b) $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & -2y + 2a^2 \end{bmatrix}$

$|H_1| = 1$

$|H_2| = -2y + 2a^2 - a^2 = -2y + a^2$

sub $y = a^2$; $-2a^2 + a^2 = -a^2$

the global maximum is $|H_1| < 0$, $|H_2| > 0$ and $d^2 f < 0$

However, in this case it is $|H_1| > 0$ and $|H_2| < 0$ so that $f(x,y;a)$ is no longer a global maximum

c) $G(a) = \frac{1}{2}(1-a^3)^2 - (1-a^3) + a(a^2)(1-a^3-1) - \frac{1}{3}(a^2)^3 + a^2(a^2)^2$

$= \frac{1}{2}(1 - 2a^3 + a^6) - 1 + a^3 - a^6 - \frac{1}{3}a^6 + a^6$

$= \frac{1}{2} - a^3 + \frac{a^6}{2} - 1 + a^3 - \frac{a^6}{3}$

$= -\frac{1}{2} + \frac{a^6}{2} - \frac{a^6}{3}$

$\frac{d\,G(a)}{da} = 3a^5 - 2a^5 = a^5$ #

d) $$\frac{\partial f(x,y;a)}{\partial a} = y(x-1) + 2ay^2$$

$$\left.\frac{\partial f(x,y;a)}{\partial a}\right|_{\substack{x^* = 1-a^3 \\ y^* = a^2}} = a^2(1-a^3-1) + 2a(a^2)^2$$

$$= -a^5 + 2a^5$$

$$= a^5$$

different answer that we derive from b

e) Hessian matrix is positive definite, as the function is convex

$$|H_1| = 1 > 0$$

$$|H_2| = -2y + a^2 > 0$$

$$-2y > -a^2$$

$$y < \frac{a^2}{2}$$

$$y \in \left(-\infty, \frac{a^2}{2}\right)$$

$$x = 1 - ay$$

if $y = -\infty$: $x = \infty$

if $y = \frac{a^2}{2}$: $x = 1 - a\left(\frac{a^2}{2}\right) = 1 - \frac{a^3}{2}$

$$x \in \left(1 - \frac{a^3}{2}, \infty\right)$$

Question 1

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.
b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.
c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.
d) What is the maximum level of profit?

a) $\pi = P \times Q - C(Q, A)$

$= P(2000 + 4\sqrt{A} - 20P) - [2Q + 1000 + A]$

$= 2000P + 4\sqrt{A}P - 20P^2 - 2(2000 + 4\sqrt{A} - 20P) - 1000 - A$

$= 2000P + 4\sqrt{A}P - 20P^2 - 4000 - 8\sqrt{A} + 40P - 1000 - A$

$= -5000 + 4\sqrt{A}P - 8\sqrt{A} - A + 2040P - 20P^2$

b) F.O.C

$\frac{\partial \pi}{\partial P} = 4A^{\frac{1}{2}} + 2040 - 40P = 0$

$4A^{\frac{1}{2}} = 40P - 2040$

$A^{\frac{1}{2}} = 10P - 510 \;;\; \frac{1}{A^{\frac{1}{2}}} = \frac{1}{10P - 510}$ —— ①

$\frac{\partial \pi}{\partial A} = 2A^{-\frac{1}{2}}P - 4A^{-\frac{1}{2}} - 1 = 0$

$A^{-\frac{1}{2}}(2P - 4) = 1$

$\frac{1}{A^{\frac{1}{2}}} = \frac{1}{2P - 4}$ —— ②

① = ② ; $\frac{1}{10P - 510} = \frac{1}{2P - 4}$

$2P - 4 = 10P - 510$

$506 = 8P$

$P = 63.25$

sub $P = 63.25$; $A^{\frac{1}{2}} = 10(63.25) - 510$

$A^{\frac{1}{2}} = 122.5$

$A = 15{,}006.25$

c) S.O.C

$$H = \begin{bmatrix} \pi_{PP} & \pi_{PA} \\ \pi_{AP} & \pi_{AA} \end{bmatrix}$$

$\pi_{PP} = -40$ | $\pi_{PA} = 2A^{-\frac{1}{2}}$

$\pi_{AP} = 2A^{-\frac{1}{2}}$ | $\pi_{AA} = -A^{-\frac{3}{2}}P + 2A^{-\frac{3}{2}}$

$$H = \begin{bmatrix} -40 & 2A^{-\frac{1}{2}} \\ 2A^{-\frac{1}{2}} & -A^{-\frac{3}{2}}P + 2A^{-\frac{3}{2}} \end{bmatrix}$$

$|H_1| = -40$

$$|H_2| = (-40)(-A^{-\frac{3}{2}}P + 2A^{-\frac{3}{2}}) - (2A^{-\frac{1}{2}})^2$$
$$= 40A^{-\frac{3}{2}}P - 80A^{-\frac{3}{2}} - 2A^{-1}$$

As $|H_1| < 0$ and $|H_2| > 0$ (with alternative sign)
H is negative definite $\Rightarrow d^2\pi < 0$ or π is concave at the solution obtained from F.O.C. (63.25, 15006.25)

π is globally concave function.

d) $\pi(P, A) = -5000 + 4\sqrt{A}P - 8\sqrt{A} - A + 2040P - 20P^2$

$= -5000 + \left(4 \times \sqrt{15006.25} \times 63.25\right) - \left(8\sqrt{15006.25}\right) - 15006.25 + (2040 \times 63.25) - 20(63.25)^2$

$= 59,025$

Question 3:

Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium
b. Suppose the government imposes unit tax on consumers in both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.
c. How much revenue can the government collect from the taxation?
d. Determine the level of t_A and t_B that maximizes government's revenue.
e. Use the second-order conditions test and show that the answer obtained in "d" is a global solution.

a) Market A

$$D = S$$
$$10 - 2Q_A = 1 + Q_A$$
$$9 = 3Q_A$$
$$Q_A = 3$$
$$P_A = 4$$

The equilibrium of Market A : $Q^* = 3 \quad P^* = 4$

Market B

$$D = S$$
$$20 - Q_B = 2 + 2Q_B$$
$$18 = 3Q_B$$
$$Q_B = 6$$
$$P_B = 14$$

The equilibrium of market B : $Q^* = 14 \quad P^* = 6$

b)

$$Q_A^d = 5 - \frac{1}{2}P_A^d$$
$$Q_A^s = P_A^s - 1$$
$$Q^d = Q^s$$
$$5 - \frac{1}{2}P_A = P_A - 1$$
$$5 - \frac{1}{2}(P_A + t_A) = P_A - 1$$
$$6 - \frac{1}{2}t_A = \frac{3}{2}P_A$$
$$P_A^s = 4 - \frac{1}{3}t_A$$
$$P_A^d = P_A^s + t_A = 4 - \frac{1}{3}t_A + t_A = 4 + \frac{2}{3}t_A$$
$$Q_A^* = P_A^s - 1 = 3 - \frac{1}{3}t_A$$

$$Q_B^s = \frac{1}{2}P_B - 1$$
$$Q_B^d = 20 - P_B^d$$
$$Q^d = Q^s$$
$$20 - P_B = \frac{1}{2}P_B - 1$$
$$20 - (P + t_B) = \frac{1}{2}P_B - 1$$
$$21 - t_B = \frac{3P_B}{2}$$
$$14 - \frac{2}{3}t_B = P_B^s$$
$$P_B^d = P_B^s + t_B = 14 + \frac{1}{3}t_B$$
$$Q_B^* = 20 - P_B^d = 6 - \frac{1}{3}t_B$$

c) Market A Tax revenue

$t_A \cdot Q_A$

$t_A\left(3 - \frac{1}{3}t_A\right) = 3t_A - \frac{1}{3}t_A^2$

Market B Tax revenue

$t_B \cdot Q_B$

$t_B\left(6 - \frac{1}{3}t_B\right) = 6t_B - \frac{1}{3}t_B^2$

d) $T = 3t_A - \frac{1}{3}t_A^2 + 6t_B - \frac{1}{3}t_B^2$

F.O.C $\frac{\partial T}{t_a} = 3 - \frac{2}{3}t_A = 0$

$-\frac{2}{3}t_A = -3$

$t_A = 4.5$

$\frac{\partial T}{\partial t_B} = 6 - \frac{2}{3}t_B = 0$

$-\frac{2}{3}t_B = -6$

$t_B = 9$

e) Find S.O.C.

$$H = \begin{bmatrix} T_{AA} & T_{AB} \\ T_{BA} & T_{BB} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}$$

$\Rightarrow |H_1| = -\frac{2}{3}$; $-\frac{2}{3} < 0$, $\forall t_A, \forall t_B$

$\Rightarrow |H_2| = \left(-\frac{2}{3}\right)^2 - 0^2 = \frac{4}{9}$; $\frac{4}{9} > 0$ $\forall t_A, \forall t_B$

$\therefore$ Therefore, as $|H_1| < 0$ and $|H_2| > 0$, then it is always negative definite, $\forall t_A, \forall t_B$

$d^2\pi < 0$, $\forall t_A, \forall t_B$

Moreover, T is globally concave function which means the answer obtained in "d" is global solution.