Chapter 4: Linear Transformation(Part 2)

1 Properties of Matrix Transformations

1.1 Compositions of Matrix Transformations

Definition 1.1. Suppose
T_A is a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^k$ and
T_B is a matrix transformation from $\mathbb{R}^k$ to $\mathbb{R}^m$.
Then the **composition** of T_B with T_A, denote by the symbol $T_B \circ T_A$ (read "T_B circle T_A"), is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})), \qquad \mathbf{x} \in \mathbb{R}^n.$$

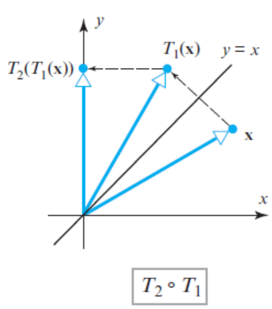
Remarks

1. This composition is itself a matrix transformation since
$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = \mathbf{B}(T_A(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x}) = (\mathbf{BA})\mathbf{x}$$
and this is expressed by the formula
$$T_B \circ T_A = T_{BA}.$$

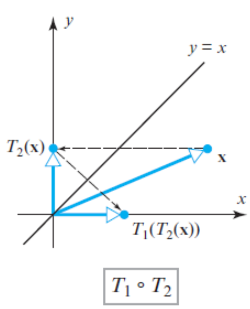
2. Suppose $T_A : \mathbb{R}^n \to \mathbb{R}^k, T_B : \mathbb{R}^k \to \mathbb{R}^\ell, T_C : \mathbb{R}^\ell \to \mathbb{R}^m$ are matrix transformation.
The composition $(T_C \circ T_B \circ T_A) : mathbbR^n \to \mathbb{R}^m$ is defined by
$$(T_C \circ T_B \circ T_A)(\mathbf{x}) = T_C(T_B(T_A(\ x))).$$

3. **Notation:** The **standard matrix** for a matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is denoted by the symbol $[T]$. So, we have
$$[T_B \circ T_A] = [T_B][T_A], \qquad [T_C \circ T_B \circ T_A] = [T_C][T_B][T_A].$$

Example 1.1. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = x$, and
let $T2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the y-axis.
Show that
$$T_1 \circ T_2 \neq T_2 \circ T_1.$$
I.e.show that the standard matrices for T_1 and T_2 do not commute



$\boxed{T_2 \circ T_1}$



$\boxed{T_1 \circ T_2}$

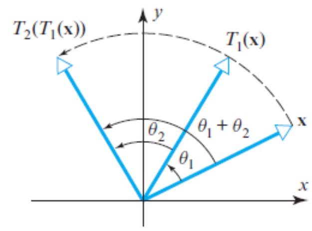
$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so $[T_2 \circ T_1] \neq [T_1 \circ T_2]$.

Example 1.2. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix operators that rotate vectors about the origin through the angles θ_1 and θ_2, respectively.

- The operation $(T_2 \circ T_1)(x) = T_2(T_1(x))$ first rotates x through the angle θ_1, then rotates $T_1(\theta_1)$ through the angle θ_2. It follows that the net effect of $T_2 \circ T_1$ is to rotate each vector in $\mathbb{R}^2$ through the angle $\theta_1 + \theta_2$
- Similarly, the net effect of $T_1 \circ T_2$ is to rotate each vector in $\mathbb{R}^2$ through the angle $\theta_2 + \theta_1$
- Hence, the composition of rotations T_1 and T_2 is commutative. I.e. $(T_2 \circ T_1)(x) = (T_1 \circ T_2)(x)$
- It can be shown directly by using the standard matrices as follows that $(T_2 \circ T_1)(x) = (T_1 \circ T_2)(x)$.



$$[T_1] = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad [T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix},$$

$$[T_2 \circ T_1] = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\begin{aligned}
[T_2][T_1] &= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -(\cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1) \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & -\sin\theta_2\sin\theta_1 + \cos\theta_2\cos\theta_1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \\
&= [T_2 \circ T_1]
\end{aligned}$$

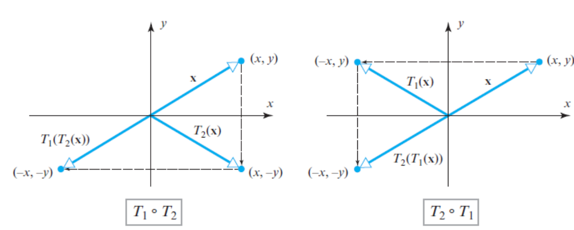
Example 1.3. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix operators that reflect vectors about the x-axis and y-axis, respectively. Determine if the composition of T_1 and T_2 is commutative.

Approach 1:

Approach 2:

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



That is, $(T_2 \circ T_1)(x, y) = (T_1 \circ T_2)(x, y) = -(x, y)$ and the composition of T_1 and T_2 is commutative.

Example 1.4. Find the standard matrix for the operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ that first rotates a vector counterclockwise about the z-axis through an angle θ, then reflects the resulting vector about the yz-plane, and then projects that vector orthogonally onto the xy-plane.

Solution: Let
T_1 be the rotation about the z-axis,
T_2 be the reflection about the yz-plane, and
T_3 be the orthogonal projection onto the xy-plane.
Then the operator T for this problem is given by

$$T =$$

Definition 1.2. A matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if T_A maps distinct vectors (points) in $\mathbb{R}^n$ into distinct vectors (points) in $\mathbb{R}^m$.
I.e., T_A is one-to-one if
- for each vector $\mathbf{b}$ in the range of $\mathbf{A}$ there is exactly one vector $\mathbf{x}$ in $\mathbb{R}^n$ such that $T_A\mathbf{x} = \mathbf{b}$;
- the equality $T_A(\mathbf{u}) = T_A(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.

Definition 1.3. Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix transformation.

- The **kernel** of T_A, denoted by $\ker(T_A)$ is the set of all vectors in $\mathbb{R}^n$ that T_A maps into $\mathbf{0}$. I.e.
$$\ker(T_A) = \text{ null space of } \mathbf{A}.$$
- The **range** of T_A, denoted by $R(T_A)$ is the set of all vectors in $\mathbb{R}^m$ that are images under this transformation of at least one vector in $\mathbb{R}^n$. I.e.
$$R(T_A) = \text{ column space of } \mathbf{A}.$$

For a linear operator $T_A : \mathbb{R}^n \to \mathbb{R}^n$, the following theorem establishes fundamental relationships between the invertibility of the standard matrix $\mathbf{A}$ and properties of T_A.

Theorem 1.1. Let $\mathbf{A}$ be an $n \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding matrix operator, then the following statements are equivalent.

(a) $\mathbf{A}$ is invertible.

(b) The kernel of T_A is $\{\mathbf{0}\}$.

(c) The range of T_A is $\mathbb{R}^n$.

(d) T_A is one-to-one.

Proof:(Exercise)
Remark:

- When $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one matrix operator, the above theorem implies that the corresponding matrix $\mathbf{A}$ is invertible and we can define **inverse operator** or **the inverse** of T_A: $T_{A^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$ that corresponds to $\mathbf{A}^{-1}$. I.e.
$$T_{A^{-1}}(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x}.$$
$$T_{A^{-1}}(\mathbf{x}) = T_A^{-1}(x).$$

Example 1.5. Show that the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors through an angle θ is one-to-one.

Example 1.6. Determine if the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ that projects onto the x-axis in the xy-plane is one-to-one.

Example 1.7. Show that the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the equations

$$\begin{aligned} w_1 &= 2x_1 + x_2 \\ w_2 &= 3x_1 + 4x_2 \end{aligned}$$

is one-to-one, and find $T^{-1}(w_1, w_2)$.

2 Application: Geometry of Matrix Operators on $\mathbb{R}^2$

The digitized image can be decomposed it into a rectangular array of pixels. Those pixels were then transformed as follows:

- The program MATLAB was used to assign coordinates and a gray level to each pixel.
- The coordinates of the pixels were transformed by matrix multiplication.
- The pixels were then assigned their original gray levels to produce the transformed picture.

In computer games a perception of motion is created by using matrices to rapidly and repeatedly transform the arrays of pixels that form the visual images.