Solution
Exercise 1

1. **BA** (a 5 x5 matrix) **ABD** (a 3x1 matrix) and **ABABD** (a 3x1 matrix)

2. $\begin{bmatrix} -6 & 21 \\ -14 & 14 \end{bmatrix}$

3.

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ -2 & -1 & 0 & -4 \\ 4 & 2 & 3 & 7 \end{array}\right] \underset{\sim}{\begin{array}{c} r_2+2r_1 \\ r_3-4r_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -6 & 2 \\ 0 & -2 & 15 & -5 \end{array}\right] \underset{\sim}{r_3+2r_1} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 3 & -1 \end{array}\right]$$

Back substitution

$$3z = -1$$
$$z = -1/3$$
$$y - 6z = 2$$
$$y = 0$$
$$x + y - 3z = 3$$
$$x = 2$$

4. a) $\left[\begin{array}{cc|c} 2 & 3 & h \\ 4 & 6 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & h \\ 0 & 0 & 7-2h \end{array}\right] \longrightarrow$ For this system to be consistent

$$7 - 2h = 0$$
$$h = 7/2$$

b) $\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 5 & h & -7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & h+15 & -17 \end{array}\right]$

This system is <u>inconsistent</u> if

$$h + 15 = 0$$
$$h = -15$$

therefore. (because $0 = -17$ is not true)

Hence, for this system to be consistent $h \neq -15$.

5.

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & \boxed{1} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The basic variable (pivot variable) is x_2

The free variable is x_1, x_3, x_4

$$AX = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ equivalent to } \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = -4x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ -4c_3 \\ c_3 \\ c_4 \end{bmatrix} \qquad c_1, c_3, c_4 \in \mathbb{R}.$$

$$\underline{X} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

FOR $A\underline{X} = \underline{b}$

Rank = 1

basic variables = x_2

free variables = x_1, x_3, x_4

$$\left[\begin{array}{cccc|c} 0 & 1 & 4 & 0 & b_1 \\ 0 & 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

$A\underline{X} = \underline{b}$ is consistent if $b_2 - 2b_1 = 0$

$$\boxed{b_2 = 2b_1}$$

$$x_2 = b_1 - 4x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} \qquad c_1, c_3, c_4 \in \mathbb{R}$$

$$\underline{X} = \underbrace{\begin{bmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix}}_{X_p} + \underbrace{c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{X_n \leftarrow (AX=0)}$$

6.

7.

Show that $A = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$ has no inverse.

If $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is *any* matrix, then $AC = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a-3c & 2b-3d \\ 0 & 0 \end{bmatrix}$. Since the (2, 2)-entry of AC is not 1, AC can never equal I for any choice of the matrix C.

Show that $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ has inverse $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

We verify that $AC = I$ and $CA = I$:

$$AC = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$CA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

8. a) m=3, 0<r<m and r<n

b) r=1, m=3, n=2; $A = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$

9.
10.

10. (a) The reduction $[A \;\; I] \to [I \;\; A^{-1}]$ is as follows:

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ -1 & 1 & -1 \end{bmatrix}$ by the matrix inversion algorithm. Of course this can be checked by verifying that $AA^{-1} = I$ and $A^{-1}A = I$.

(b)

We try the algorithm on A to get:

$$[A \;\; I] = \begin{bmatrix} 1 & 3 & -2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 8 & -8 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 2 & -4 & -2 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -4 & 2 & 1 \end{bmatrix}$$

Since A will never be transformed to the identity matrix by elementary row operations, A is not invertible.

11 (a)

$$A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}$$

(b) U has 4 nonzero entries on the diagonal
$\Rightarrow A$ has 4 nonzero pivots
$\Rightarrow$ Gauss-Jordan will work
$\Rightarrow A^{-1}$ exists

(c) If the last diagonal entry of U was zero $\Rightarrow A_{44} = 1 + 9 + 9 = 19$.

(d)

$$P = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \Rightarrow PA \text{ has reversed rows \& } AP \text{ has reversed columns.}$$

6.

(6) a) $\underline{x} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$; $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$

b) $\underline{x} = \begin{bmatrix} a-3b \\ 0 \\ b \\ 0 \end{bmatrix}$

9.

The lower reduction to row-echelon form is as follows:

$$\begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 8 & 2 & 4 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 8 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $A = LU$ where $L = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & 8 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

10.

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 4 & 2 \\ 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = P_2P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$PA = \begin{bmatrix} \text{[redacted]} & 4 & 2 \\ & -1 & 4 \\ & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & \text{[redacted]} & 3 \\ 0 & & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & \text{[redacted]} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U.$$

Ience $PA = LU$ where $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.