System Equations Estimation Methods

System Equations Models

Seemingly Unrelated (SUR) Model

Simultaneous Equations Model

- Limited Information Estimation Methods (Single Equation Estimation Methods)
  - ILS, 2SLS, & (LIML)
- Full Information Estimation Methods (System Equation Estimation Methods)
  - 3SLS, I3SLS, & (FIML)

Seemingly Unrelated Regression (SUR) Models

The models:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{pmatrix}$$

or $y = X\beta + \varepsilon$

where

$$E[\varepsilon] = 0, \mathrm{var}(\varepsilon) = V = \begin{pmatrix} \sigma_{11} I & \sigma_{12} I & \cdots & \sigma_{1m} I \\ \sigma_{21} I & \sigma_{22} I & \cdots & \sigma_{2m} I \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} I & \sigma_{m2} I & \cdots & \sigma_{mm} I \end{pmatrix}$$

Seemingly Unrelated Regression (SUR) Models

Example: System of two equations:

$$y_{1t} = \beta_{10} + \beta_{11} x_{1t} + \beta_{12} x_{2t} + u_{1t}$$
$$y_{2t} = \beta_{20} + \beta_{22} x_{2t} + \beta_{23} x_{3t} + u_{2t}$$

If there exists relationship of the error terms across equations, i.e. $E(u_{1i}, u_{2i}) \neq 0$.

Ignorance of this problem by using single equation estimation method (or separate OLS) will lead to less efficient estimators.

Estimation of SUR Models

Two-step system estimation or FGLS

1. Estimate Var-Cov matrix $\hat{V}$
2. Estimate all parameters β_j using FGLS.

$$\hat{\beta}_{SUR} = \hat{\beta}_{FGLS} = \left(X'\hat{V}^{-1}X \right)^{-1} X'\hat{V}^{-1}y$$

Then, system equation estimation method will provide a more asymptotically efficient estimators.

However, if there exists specification error problem in any equations, the problem will spread through out the whole system.

Simultaneous Equations Model

More than one equation.

Jointly dependent or endogenous variables.

General Structural Form model can be stated as:

$$\gamma_{11}y_{t1} + \gamma_{21}y_{t2} + \cdots + \gamma_{m1}y_{tm} + \beta_{11}x_{t1} + \cdots + \beta_{k1}x_{tk} = \varepsilon_{t1}$$

$$\gamma_{12}y_{t1} + \gamma_{22}y_{t2} + \cdots + \gamma_{m2}y_{tm} + \beta_{12}x_{t1} + \cdots + \beta_{k2}x_{tk} = \varepsilon_{t2}$$

$$\vdots$$

$$\gamma_{1m}y_{t1} + \gamma_{2m}y_{t2} + \cdots + \gamma_{mm}y_{tm} + \beta_{1m}x_{t1} + \cdots + \beta_{km}x_{tk} = \varepsilon_{tm}$$

Simultaneous Equations Model

Structural Form model:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}_t \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mm} \end{bmatrix} +$$

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}_t \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{bmatrix} = \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \cdots & \varepsilon_{mt} \end{bmatrix}$$

or $$\mathrm{y}_t'\Gamma + \mathrm{x}_t'\mathrm{B} = \varepsilon_t'$$

Simultaneous Equations Model

Two endogenous variables models:

$$y_{1t} = \beta_{10} + \beta_{12} y_{2t} + \gamma_{11} x_{1t} + u_{1t}$$

$$y_{2t} = \beta_{20} + \beta_{21} y_{1t} + \gamma_{21} x_{1t} + u_{2t}$$

<u>Example</u>

Keynesian Income Determination Model

Consumption Function $C_t = \beta_0 + \beta_1 Y_t + u_{1t}$

Income Identity $Y_t = C_t + I_t$

System Equations Model

Simultaneous Bias occurs when $Cov(Y_t, u_t) \neq 0$

OLS estimators will be biased, inconsistent, and inefficient.

$E(\hat{\beta}) \neq \beta$ Biased

$p\lim(\hat{\beta}) \neq \beta$ Inconsistent

System Equations Model

Structural Form Model

shows structural relationship between endogenous variables and exogenous variables & lagged endogenous variables.

$$C_t = \beta_0 + \beta_1 Y_t + u_{1t}$$

$$Y_t = C_t + I_t$$

Reduced Form Model

shows relationship only between endogenous variables and exogenous variables.

$$C_t = \pi_0 + \pi_1 I_t + w_{1t}$$

$$Y_t = \pi_2 + \pi_3 I_t + w_{2t}$$

where

$$\pi_0 = \frac{\beta_0}{1-\beta_1} \qquad \pi_1 = \frac{\beta_1}{1-\beta_1}$$

$$\pi_2 = \frac{\beta_0}{1-\beta_1} \qquad \pi_3 = \frac{1}{1-\beta_1}$$

$$w_{1t} = \frac{u_{1t}}{1-\beta_1} \qquad w_{2t} = \frac{u_{2t}}{1-\beta_1}$$

Single Equation Estimation Methods

<u>Indirect Least Squares (ILS)</u>

- Estimate Reduced form model using OLS
- Solve for estimated β

Structural Form Model:

$$Y\Gamma + XB = E$$

Reduced Form Model:

$$Y = X\Pi + V$$

ILS: 1. Estimate Reduced Form $\hat{\Pi} = \left(X'X\right)^{-1} X'Y$

2. Solve for B $\hat{B}_{ILS} = -\hat{\Pi}\Gamma$

Single Equation Estimation Methods

Two-Stage Least Squares (2SLS)

1. Estimate Reduced form model using OLS
   Predict endogenous variable $(\hat{Y})$
2. Use $\hat{Y}$ as Instrumental Variables (IV) for lagged endogenous variables instead of using actual Y, then, obtain matrix $\hat{Z}$, then estimate model using IV technique

$$\hat{B} = \left(\hat{Z}'\hat{Z}\right)^{-1}\hat{Z}'Y$$

System Equations Estimation Methods

<u>Three-Stage Least Squares (3SLS)</u>

If there exists relationship of the error terms across equations, system equation estimation will lead to more asymptotically efficient estimators.

2SLS + construct estimated V matrix

3-stage perform FGLS using estimated S matrix

$$\hat{\mathrm{B}} = \left[\hat{\mathrm{Z}}' \left(\hat{V}^{-1} \otimes I \right) \hat{\mathrm{Z}} \right]^{-1} \hat{\mathrm{Z}}' \left(\hat{V}^{-1} \otimes I \right) y$$

OLS, ILS, 2SLS, 3SLS

<u>Single Equation Estimation Methods</u>

OLS: biased & inefficient

ILS: biased & Asymptotically efficient

2SLS: biased & Asymptotically efficient

<u>System Equation Estimation Methods</u>

3SLS: biased & More Asymptotically efficient

but if there exists specification error in one or any equation(s), the specification error problem will spread through all equations in the system.