Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E\left[mX_a\right] \end{aligned}$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

$$p_s = \pi_s m_s 1$$

$$m_s = \frac{\delta u'(C_s^*)}{u'(C_0^*)}$$

$$m_s = \delta \frac{C_0^*}{C_s^*}$$

so, $$p_s = \delta \pi_s \frac{C_0^*}{C_s^*}$$

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3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E\left[m_{01} R_s\right]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U\left(C_0\right) + \delta E\left[U\left(\tilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U(C) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

Since $m_{01} = \frac{\delta U'(C_1)}{U'(C_0)} = \delta\left(\frac{C_1}{C_0}\right)^{\gamma-1}$

FOC: $1 = \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} R_s$

$0 = \pi_s \delta(\gamma-1)\left(\frac{C_s}{C_0}\right)^{\gamma-2} R_s \, d\left(\frac{C_s}{C_0}\right) + \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s$

Rearrange: $\frac{R_s}{C_s/C_0} \frac{d(C_s/C_0)}{dR_s} = \frac{1}{1-\gamma}$

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4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.[16] There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

a) Let $p = \begin{bmatrix} 1/1.05 \\ 6 \end{bmatrix}$

$$X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}$$

Then, $[p_1 \quad p_2] = p' X^{-1}$

$$= \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix} = [0.2476 \quad 0.7048]$$

Finally, $\hat{\pi}_1 = p_1 R_f = 0.26$ #

$\hat{\pi}_2 = p_2 R_f = 0.74$ #

b) $\pi_1 = \pi_2 = 0.5$

$$m_1 = \frac{p_1}{\pi_1} = \frac{0.2476}{0.5} = 0.495 \text{ \#}$$

$$m_2 = \frac{p_2}{\pi_2} = \frac{0.7048}{0.5} = 1.410 \text{ \#}$$

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6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

a)

$$0 = E[m(R_i - R_f)]$$
$$= E[(a + bR_m)(R_i - R_f)]$$
$$= aE[R_i] - aR_f + b(E[R_m]E[R_i]) - bR_f E[R_m]$$
$$= a(E[R_i] - R_f) + b(E[R_m]E[R_i] - R_f E[R_m] + Cov[R_m, R_i])$$
$$= (E[R_i] - R_f)(a + bE[R_m]) + b\,Cov[R_m, R_i]$$

Rearrange:

$$E[R_i] - R_f = \frac{-b\,Cov[R_m, R_i]}{a + bE[R_m]}$$
$$= -\frac{Cov[R_m, R_i]}{\sigma_m^2}\left(\frac{b\sigma_m^2}{a + bE[R_m]}\right)$$
$$= -\beta_i\left(\frac{b\sigma_m^2}{a + bE[R_m]}\right)$$

$$\therefore \gamma = -\frac{b\sigma_m^2}{a + bE[R_m]} \quad \#$$

b)

$$\frac{1}{R_f} = E[m] = E[a + bR_m]$$
$$a = \frac{1}{R_f} - bE[R_m]$$

For the market port,

$$1 = E[(a + bR_m)R_m]$$
$$= aE[R_m] + bE[R_m^2]$$
$$= aE[R_m] + b(\sigma_m^2 + E[R_m]^2)$$

Plug a into the equation

$$1 = \left(\frac{1}{R_f} - bE[R_m]\right)E[R_m] + b(\sigma_m^2 + E[R_m]^2)$$
$$= \frac{E[R_m]}{R_f} + b\sigma_m^2$$

so, $b = -\frac{E[R_m] - R_f}{R_f \sigma_m^2}$ #, $a = \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f)}{R_f \sigma_m^2}$ #

c)

$$\gamma = E[R_m] - R_f$$

$$a + bE[R_m] = \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f) - E[R_m](E[R_m] - R_f)}{R_f \sigma_m^2}$$
$$= \frac{1}{R_f}$$

$$\gamma = \frac{-b\sigma_m^2}{a + bE[R_m]}$$
$$= \frac{E[R_m] - R_f}{\cancel{R_f}\cancel{\sigma_m^2}} \cancel{\sigma_m^2}\cancel{R_f} = E[R_m] - R_f \quad \#$$