Fundamentals of Mathematical Proofs: III

TU152: Fundamental Mathematics

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Indirect Proof

1. **Method of Proof by Contradiction**

2. **Method of Proof by Contrapositive**

Proof by Contradiction

Method of Proof by Contradiction

1. Suppose the statement to be proved is false. That is, suppose that the negation of the statement is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Example: If a man accused of holding up a bank can prove that he was some place else at the time the crime was committed, he will certainly be acquitted. The logic of his defense is as follows:

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Suppose I did commit the crime.  Then at the time of the crime,
I would have had to be at the scene of the crime.  In fact, at
the time of the crime I was in a meeting with 20 people far from
the crime scene, as they will testify.  This contradicts the
assumption that I committed the crime since it is impossible to
be in two places at one time.  Hence that assumption is false.
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Proof by Contradiction

Example: Use proof by contradiction to show that

There is no greatest integer.

Proof:

Proof by Contradiction

Example: Use proof by contradiction to show the following statement is true.

There is no integer that is both even and odd.

Proof:

Proof by Contradiction

Example: Use proof by contradiction to show the following statement is true.

> The sum of any rational number and any irrational number is irrational.

Proof:

Method of Proof by Contraposition

To prove a statement by contraposition, you take the contrapositive of the statement, prove the contrapositive by a direct proof, and conclude that the original statement is true.

Method of Proof by Contraposition

1. Express the statement to be proved in the form

$$\forall x \in D, \text{ if } P(x), \text{ then } Q(x).$$

(This step may be done mentally.)

2. Rewrite this statement in the contrapositive form

$$\forall x \in D, \text{ if } Q(x) \text{ is false, then } P(x) \text{ is false.}$$

(This step may also be done mentally.)

3. Prove the contrapositive by a **direct proof**.
   (i) Suppose x is a (particular but arbitrarily chosen) element of D such that $Q(x)$ is false.
   (ii) Show that $P(x)$ is false.

Proof by Contraposition

Example: Use proof by *contraposition* to show the following statement is true.

For all integers n, if n^2 is even, then n is even.

Proof:

Relation between Proof by Contradiction and Proof by Contraposition

- Proof by contrapositon can only be used to prove the statements that are *universal* and *conditional*, i.e.

$$\forall x \in D, \text{ if } P(x), \text{ then } Q(x).$$

- Any proof by *contraposition* can be done by *contradiction*. I.e. Consider

$$\forall x \in D, \text{ if } P(x), \text{ then } Q(x).$$

  - Using proof by contraposition: $\forall x \in D, \text{ if } Q(x) \text{ is false, then } P(x) \text{ is false.}$

$$\boxed{\text{Suppose for any } x \in D \text{, such that } \sim Q(x)} \rightsquigarrow\rightsquigarrow\rightsquigarrow \boxed{\sim P(x)}$$

  - Using proof by contradiction:

$$\boxed{\text{Suppose } \exists x \in D \text{, such that } P(x) \text{ and } \sim Q(x)} \rightsquigarrow\rightsquigarrow\rightsquigarrow \boxed{\underbrace{P(x) \text{ and } \sim P(x)}_{\text{CONTRADICTION!}}}$$

- Some statements can be proved by contradiction but not by contraposition, such as the statement

"$\sqrt{2}$ is irrational."

- In proof by contradiction, it may be difficult to know what contradiction to head for (in the proof by contraposition, you know exactly what conclusion you need to show, namely the negation of the hypothesis).

Example: Use proof by *contradiction* to show the following statement is true.

For all integers n, if n^2 is even, then n is even.

Proof:

Example: Proof that

$\sqrt{2}$ is irrational.

Proof: