Relation and function: some applications in economics

EE320

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Today

- Review some important concepts of relation and function
- Discuss about two major classes of function: linear vs non-linear
- Showing the application that we can work with linear and non-linear function in economics.
- If time permits, I'll just go on talking about solving the 2x2 system of equations.

Relation

- Mathematically, a set-to-set mapping operation that produces a set of ordered pairs.
- Ordered pairs are usually generated by using the concept of “*Cartesian product*”.
  - Cartesian product = summarizes all possible ways that sets can be mapped into each other.
- Any ordered pairs, to be grouped into a certain set of relation, they must have some properties/characteristics commonly shared.

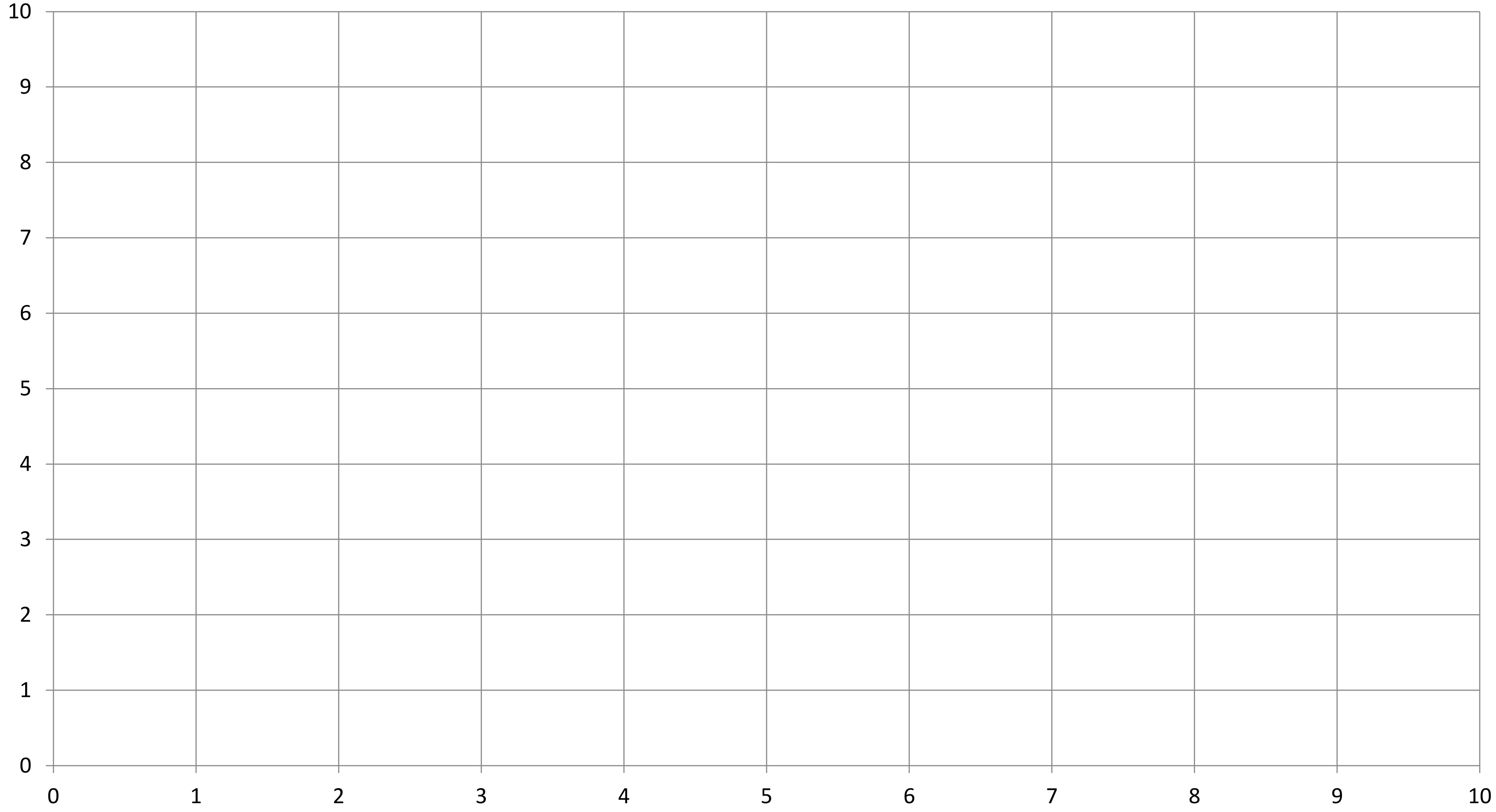
Relation

- Example,
  - A is a set of integer values between 0 and 10, representing price/unit of output
  - B is a set of integer values between 0 and 10, representing quantity of output.

  A = { 0 , 1, 2 , 3, … ,10}

  B = { 0 , 1, 2 , 3, … ,10 }
- Cartesian products from A to B is 11 x 11 elements of ordered pairs.

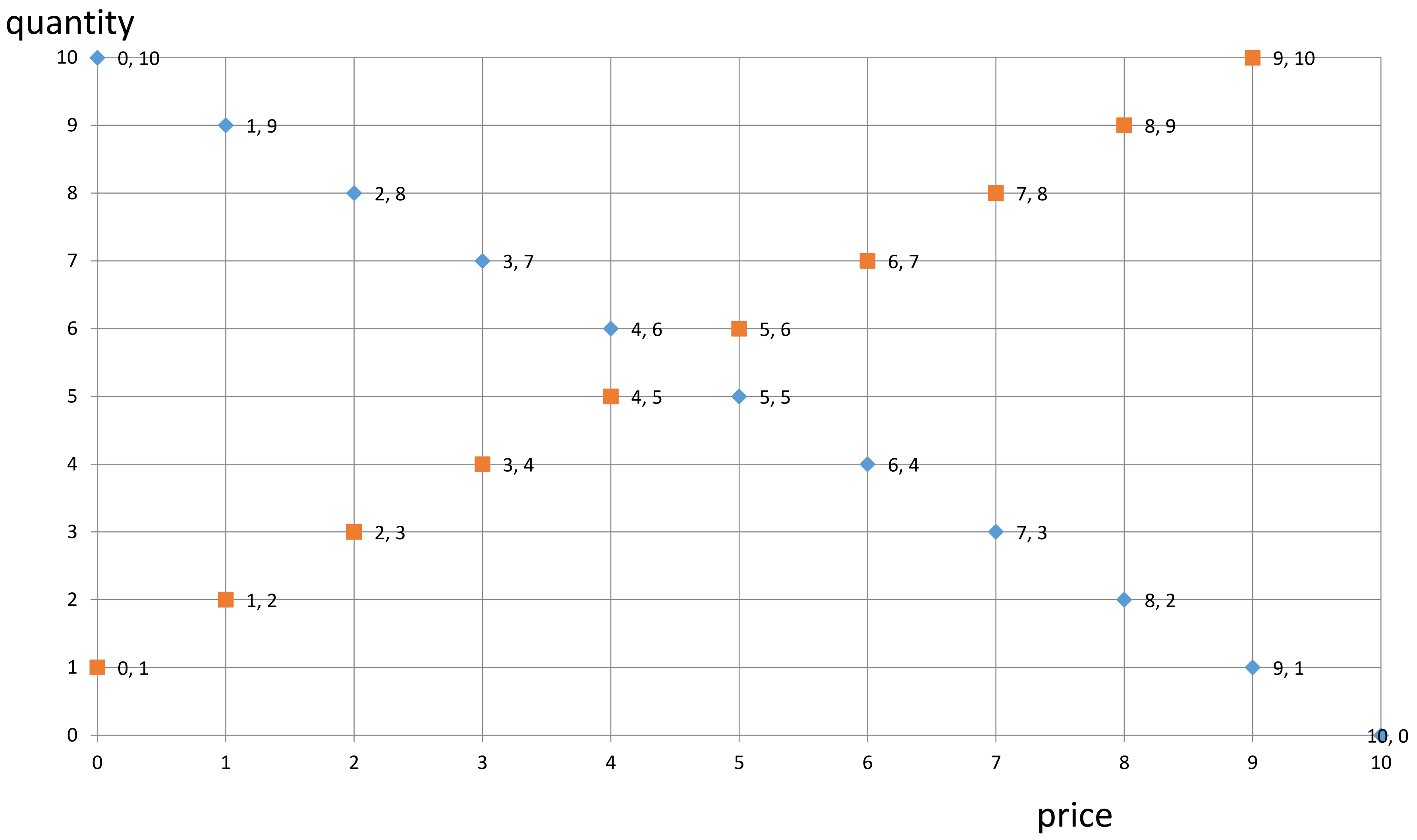
Graphically illustrated for the Cartesian products



Relation

- As said before, only a subset of Cartesian products with some specification rules applied.
- Two selected relations
  - R1 = { (0,10) , (1,9), (2,8), (3,7), (4,6), (5,5), (6,4), (7,3), (8,2), (9,1), (10,0) }.
  - R2 = { (0,1), (1,2) , (2,3), (3,4), (4,5), (5,6), (6,7), (7,8), (8,9), (9,10) }.

Graphically illustrated the two selected relations on Cartesian products x-y plane.



Alternative representation using mathematical equation

- $R1 = \{(x, y) \in A \times B | y = 10 - x\}$
  - R1 = relation that represent demand schedule

- $R2 = \{(x, y) \in A \times B | y = x + 1\}$
  - R1 = relation that represent supply schedule

- Representing mathematical relation by listing down all the ordered-pairs under the said relation.
- Graphical illustration helps when mapping is only for two sets. When mapping for the relation is for more than two sets, it's hard to visualize.
- Most of the time, we will express mathematical relation using mathematical equation.

Relation

- Type of relations.
  - one-to-many: a point in domain set is mapped into a number of points in anti-domain (range) set.
  - many-to-one: several points in the domain set are mapped into a single value in the anti-domain (range) set.
  - one-to-one (injective): each point in the domain set is mapped to different value in the anti-domain set.
  - See example in note!

Function

- Function is the only two types of relation: *many-to-one and one-to-one*.
- Operation on function
  - Inverse
  - Composite
  - Algebraic

Two major classes of function

- Linear function
- Non-linear function
  - Polynomial
  - Exponential
  - Logarithm

Linear function

- Functional form: $y = mX + b$.
- "m" is the slope of linear function = change in Y over change in X = "rise" *over* "run".
- b is the value of y-intercept. That is, the value of Y when $X = 0$.

Property of linear function

- (i) the value of “m” tells us about the shape of linear function.
  - $m > 0$ == upward sloping
  - $m < 0$ == downward sloping
  - $m = 0$ == Horizontal line that is parallel to x-axis.
- (ii) the large value of “m” measured in absolute term, the stepper the line is.
  - So, the extreme case for the value of slope is when “m” is infinity. That is, we have vertical line parallel to y-axis.
- (iii) Two lines are parallel to each other when $m1 = m2$. (two lines that never cross on x-y axis.)
- (iv) Two lines are perpendicular/orthogonal

  when $m1*m2 = -1$

- <u>Example:</u> plotting linear equations
  - L1: $y = -3x + 4$;
  - L2: $3x + y = 7$;
  - L3: $y = (1/3)x - 5$;
  - L4: $y = 5$.
  - L5: $x = 7$.

Answer: L1 // L2 (slope = -3) <u>*and*</u> L1 is orthogonal to L3. (and L2 as well.)

Application: a simple linear economic model for Break-even analysis

- A representative firm is operating in a perfectly competitive market. Assume that market price is equal to \$7. Suppose that firm faces a constant value of AVC (average variable cost) equal to \$3. Additionally, firm must pay for the fixed cost of the operation equal to \$100. (Suppose the following notations: Q is quantity of output, P is price per unit.)
  a) Find the level of output that results in break-even situation of this firm.
  b) Would the answer as in “a” change if the fixed cost is change to \$150 and \$200, respectively. Can we drawn any implications from this question?

Application: a simple linear economic model for Break-even analysis

- Answer
  - TC = TFC + TVC = 100 + AVC*Q = 100 + 3*Q
  - TR = P*Q = 7*Q.
- So, break-even output is "Q" such that TC = TR.
- That is, Q* = 25.
- When fixed cost increases, you need to produce more.
  - TFC = 150 → Q* = 37.5 units.
  - TFC = 200 → Q* = 50 units.

Application: Understanding "Ceteris Paribus"

- One usually sees in textbooks/examples that individual demand function takes the form of $Q_x = a - bP_x$. Economically, such form is actually being simplified from a true underlying structural relationship that may in fact be given by $QD_x = 100 - 0.2 * P_x + 0.2 * P_y - 0.01 * P_z + 0.002 * \text{Income}$
  a. Calculate quantity demanded for "x" if $P_x = 10$, $P_y = 10$, $P_z = 100$ and Income = 1,000.
  b. Suppose all everything is fixed to the numeric values given in "a", except Px that is allowed to be varied. Plot demand for x on p-q diagram, where P is vertical axis and Q is horizontal axis.
  c. How would you prove the pairwise relationship between goods x, good y and good z. (no need to think about the relationship between good y and good z.)
  d. Is X a normal goods?

Application: Understanding "Ceteris Paribus"

- Answer
  - $Q_x^D = 100 \ - 0.2 * P_x + 0.2 * P_y - 0.01 * P_z + 0.002 * Income$
  - The form of demand equation "Qx =a-bPx" is in fact a simplified version of the above equation. This simplification captures the concept of ceteris paribus. That's we keep all other factors as constant, while allowing only "Px" to be varied!.
  - In the example,
    - $Q_x^D = 100 \ - 0.2 * P_x + 0.2 * (10) - 0.01 * (100) + 0.002 * (1000)$
    - $Q_x^D = 103 \ - 0.2 * Px$

Application: Understanding "Ceteris Paribus"

- Rewrite the equation in p-equal form, we obtain that
  - $P_x = 515 \ -5Q_x^D$
  - Based on this equation, this consumer would stay in the market only when price is less than or equal to $515. That's the highest level of price that this consumer would be wiling to pay is $515.
  - Q-intercept is 103. That's when you were given this product for free, this consumer would choose to consume equal to 103 units.

Application: Understanding "Ceteris Paribus"

- To understand the relationship between goods x and good y, i.e. what types of good they are, we can just look at how p-intercept of the p-equal equation changes when price of y increases.
- As you can see "515" is obtained when we fix Py = 10, along with others at their given values.
- If x and y are substitute product, an increase in price of y should shift up the demand for X. That's we should expect to get new p-intercept value that is larger than 515 when Py has increased.

Application: Understanding "Ceteris Paribus"

- As the demand equation is linear function in all the arguments on the right, an easier approach is to directly check it from the sign of the coefficients associated to each variable.
  - $P_x \uparrow \Longrightarrow Q_x^D \downarrow$ (Law of demand)
  - $P_y \uparrow \Longrightarrow Q_x^D \uparrow$ (Substitution product)
  - $P_z \uparrow \Longrightarrow Q_x^D \downarrow$ (Complement product)
  - $Income \uparrow \Longrightarrow Q_x^D \uparrow$ (normal goods)
- This is not warranted under non-linear function. However, we do have ways to check for the relationship. But we need to study the concept of partial derivative concept first.

Application:
Slope V.S. Elasticity of supply

- Suppose that the individual supply is equal to

  $P = 4 + Q$.

  - Calculate the value of elasticity of supply when price changes from \$7 to \$8. Use the point-formula. Compare the result with an alternative case when price instead changes from \$8 to \$7.
  - Redo the exercise, but now use arc-elasticity formula for the calculation of both cases. Compare the results.
  - How about the value of elasticity of supply when $p = \$6$.
  - Show in general case for $P = a + bQ$ (such that $b>0$) that elasticity of supply is always greater than one if $a > 0$, equal to 1 if $a=0$, and less than one if $a < 0$.

Slope v.s. Elasticity

- Are they the same concept?
- What’s the formula for Elasticity? Does it relate to slope?

Slope and Elasticity

- Suppose $y = f(x)$
  - $Slope = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$
  - Point- Elasticity = $\frac{\%\,\Delta y}{\%\,\Delta x} = \frac{\left(\frac{\Delta y}{y_0}\right)}{\left(\frac{\Delta x}{x_0}\right)} = \frac{y_1 - y_0}{x_1 - x_0} * \frac{x_0}{y_0} = slope * \left(\frac{x}{y}\right)$
  - Arc-elasticity = $\frac{\left(\frac{\Delta y}{\frac{1}{2}(y_1 + y_0)}\right)}{\left(\frac{\Delta x}{\frac{1}{2}(x_1 + x_0)}\right)} = \frac{\Delta y}{\Delta x} * \frac{x_1 + x_0}{y_1 + y_0}$

Application:
Slope V.S. Elasticity of supply

- P0 = $7 , Q0 = 3 and P1 =8, Q1 = 4
  - Elasticity = 1*(7/3) = 7/3.
- P0 =$8, Q0 = 4 and P1 = $7 , Q1 = 3
  - Elasticity = 1*(8/4) = 2.
- Arc elasticity = 1*(15/7) = 15/7.
- P = 6 , Q = 2.
  - Elasticity = slope * P/Q = (1) * (6/2) = 3.

Application:
Slope V.S. Elasticity of supply

- Elasticity of supply $= \frac{1}{b} * \frac{(a+bQ)}{Q}$

$$= \frac{a}{b} * \frac{1}{Q} + 1$$

According to law of supply, b > 0.

Also, by the restriction of domain, Q > 0.

$\frac{a}{b} * \frac{1}{Q} > 0$ if a > 0 ➔ Elasticity > 1.

$\frac{a}{b} * \frac{1}{Q} < 0$ if a < 0 ➔ Elasticity < 1.

Application:
Slope V.S. Elasticity of demand

- Suppose that market demand is given by

  P = 10 - 2Q.

  - Calculate the value of elasticity of demand at P = 5.
  - Show in the general case for P = a – b*Q that the absolute value of elasticity of demand is equal to $\frac{p}{a-p}$.
  - Using the above expression to show that elasticity of demand is equal to 1 at the mid-point of demand curve. Further show that when price is above (below) the mid-point, demand is elastic (inelastic).

Application:
Slope V.S. Elasticity of demand

- Slope = -2.
- BUT, this is not the slope of what we want. We want the slope of equation in q-equal form.
- By the property of linear equation, slope of the q-equal equation is -½.
- Elasticity = -½*(5/2.5) = -1 (unit elasticity!)

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Slope V.S. Elasticity of demand

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Application:
Slope V.S. Elasticity of demand

- From the formula,

$$\left|\varepsilon^d\right| = \frac{1}{b} * \frac{p}{Q}$$

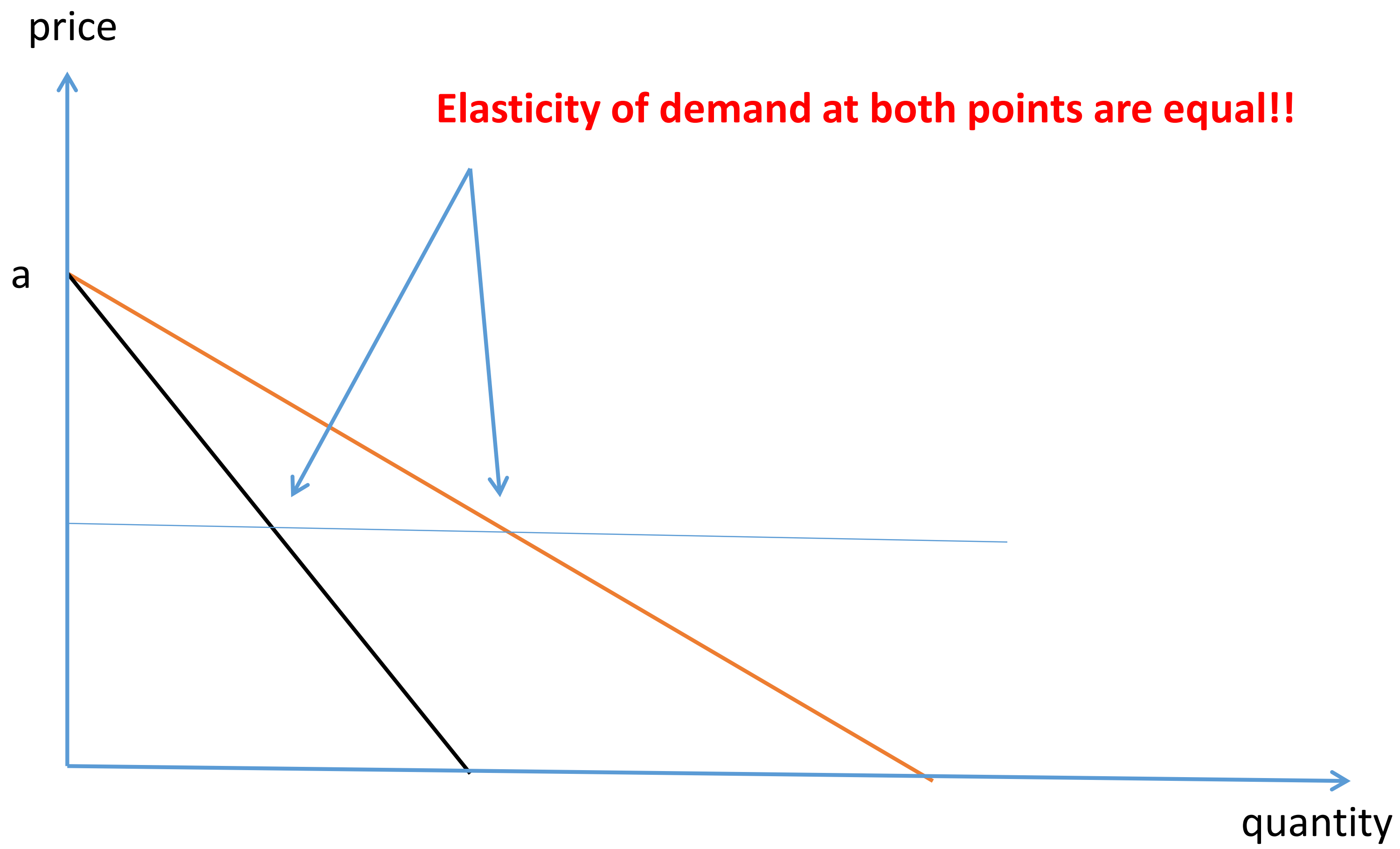
$$= \frac{1}{b} * \frac{p}{\frac{(a-p)}{b}}$$

$$= \frac{p}{a-p}$$

Application:
Slope V.S. Elasticity of demand

- Implication:
  - any two demand equations that have the same intercept value on p-axis (“a”) would have the same value of elasticity of demand when evaluated at the same price. This is despite that both demand curves have different level of slope.

Application:
Slope V.S. Elasticity of demand



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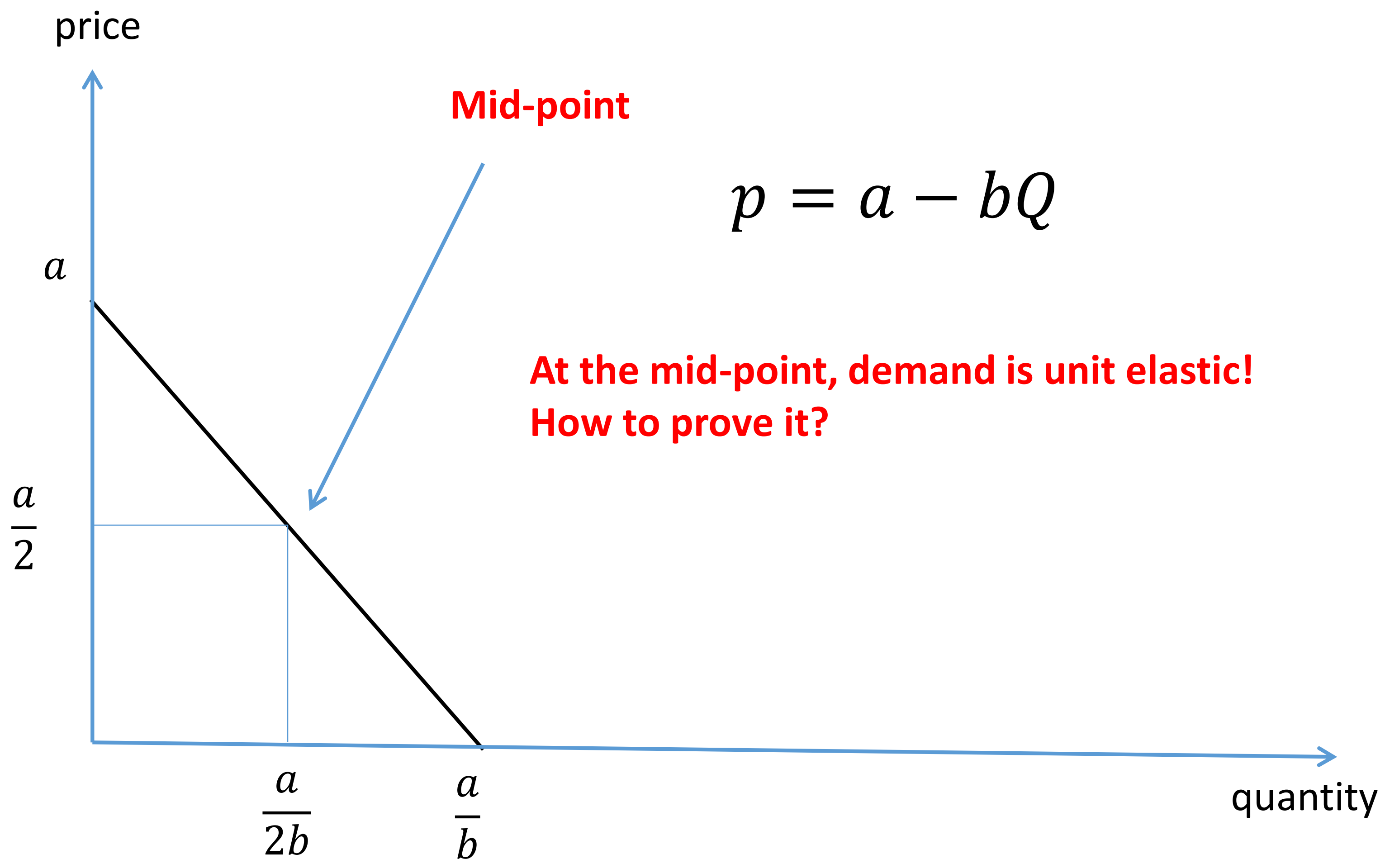
Application:
Slope V.S. Elasticity of demand

- First, what is the co-ordinate of the mid-point?
- Mid-point = $(p,q) = (\ a/2\ ,\ \frac{1}{2}(a/b)\ )$

Application:
Slope V.S. Elasticity of demand



$$p = a - bQ$$

At the mid-point, demand is unit elastic!
How to prove it?

Application:
Slope V.S. Elasticity of demand

- From the formula,

$$\frac{p}{a-p} = 1 \; when \; p = \frac{a}{2}$$

- Note further that, $p > \frac{a}{2} \Rightarrow \frac{p}{a-p} > 1$. Thus, above the mid-point, demand is elastic.

Non-linear function: polynomial

- Functional form:
  - $y = a_0 + a_1 x + a_2 x^2 + \ \dots + a_n x^n$
- n = 1 ➔ Linear function
- n = 2 ➔ Quadratic function
- n = 3 ➔ Cubic function

Quadratic function

- Functional for: $y = c + bx + ax^2$
  - a > 0 ➔ u-shape
  - a < 0 ➔ inverted u-shape
- Extreme point x* = -b/2a. Type of extreme point depends on shape of parabola.

Application

- **Example**: (Revenue function in the quadratic form) Suppose that demand function is given by: $P = 200 - Q$. P is the price per unit, and Q is the amount of goods purchased.
  - Find the revenue function.
  - Plot graph for the revenue function, and locate some important points.
  - What is the level of revenue-maximizing output?
  - Suppose further that cost function can be given by $C(Q) = 200Q+100$, find the level of break-even output.
  - What is the level of profit-maximizing output/price?

Application

- TR = P*Q
- Depending on market structure.
  - Fixed price
  - Varied price, captured by market demand equation
- $R = (200 - Q)Q = 200Q - Q^2$

Application

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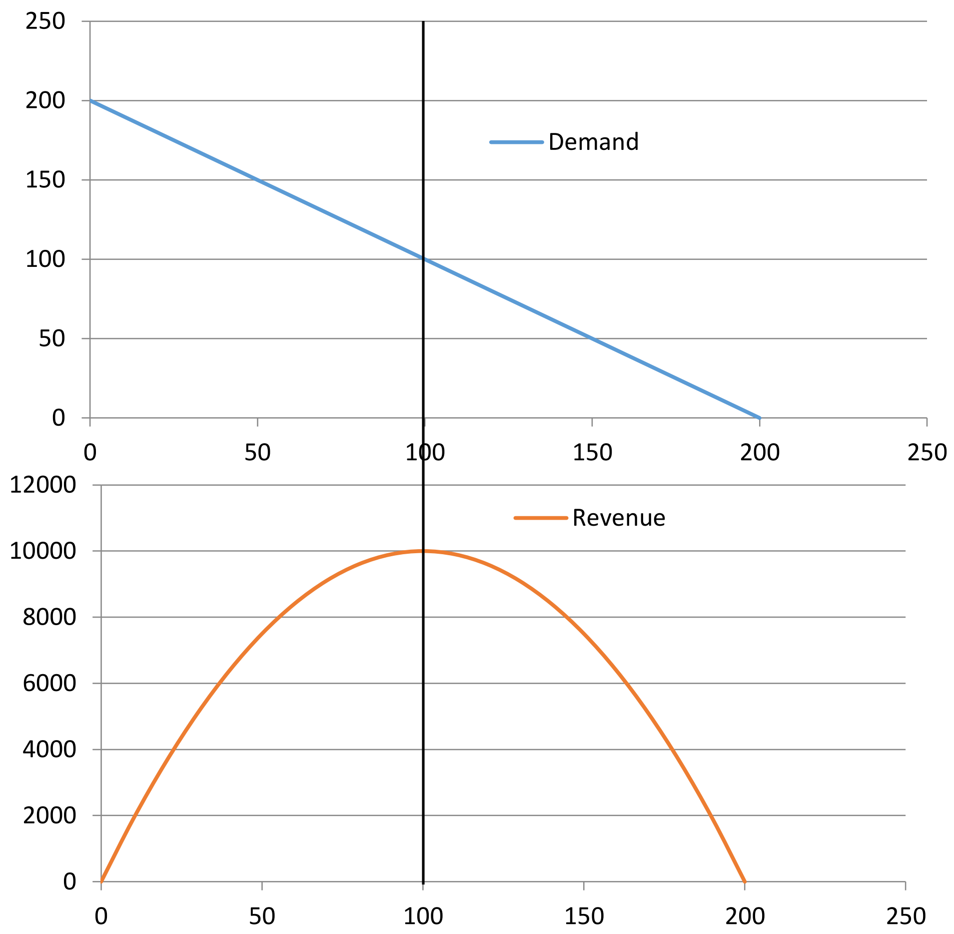
Application

- $R = 200Q - Q^2$
  - Intercepts Q = 0 and Q = 200.
  - b = -1 ➔ maximum point! (an inverted U-shape)
  - $Q^* = -\frac{200}{(2)(-1)} = 100.$
  - $P^* = -\frac{200}{(2)(-1)} = 100.$
  - Maximized revenue is 10,000 (= 100 x 100).



Notice sth from this coordinate!

(p = 100, Q = 100) on the demand curve

“Unit Elastic demand”