EE325

Answer HW 1

1

a) $\sum_{i=1}^{5} (a+bx_i) = \sum_{i=1}^{5} a + \sum_{i=1}^{5} bx_i = 5a + b\sum_{i=1}^{5} x_i$

b) $\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + \ldots + 10^2$

c) $\sum_{y=0}^{5} f(x+y) = f(x+0) + f(x+1) + \ldots + f(x+5)$

d) $\sum_{x=1}^{2}\sum_{y=2}^{3} (2x+y) = \sum_{x=1}^{2} ((2x+2) + (2x+3))$

$= (2(1)+2) + (2(2)+2) + (2(1)+3) + (2(2)+3)$

$= 22$

2

a) $\sum f(x) = 1$

$= 0.5b + b + 2.25b + 2b + 1.5b + 0.5b + 0.25b$

$= 8b$

$\frac{1}{8} = b$

b) $P(X \leq 2) = 1 - 0.75b = 1 - 0.75(\frac{1}{8})$

c) $P(-1 \leq X \leq 3) = 1 - \left(P(X > 3) + P(X < -1)\right)$

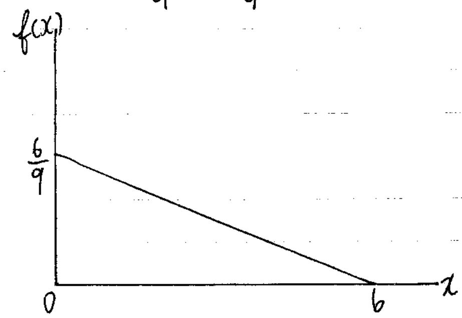
$= 1 - (0.25b + 0.5b)$

$= 1 - 0.30b = 1 - 0.30(\frac{1}{8})$

d) $P(X \geq 1) = 1 - P(X < 1)$
$= 1 - (0.5b + b + 2.25b)$
$= 1 - 3.75b$
$= 1 - 3.75\left(\frac{1}{8}\right)$

3 $f(x) = -\frac{1}{9}x + \frac{6}{9}$, $0 \leq x \leq 3$

a)



b) $P(1 \leq X \leq 3)$

$\int_1^3 f(x)\,dx$

$\int_1^3 \left(-\frac{1}{9}x + \frac{6}{9}\right) dx$

$\left.\frac{-x^2}{18} + \frac{6x}{9}\right|_1^3 = \frac{8}{9}$

(c) $P(X \geqslant 2) = \int_2^3 f(x)\, dx$

$$= \int_2^3 \left(-\frac{1}{9}x + \frac{6}{9}\right) dx$$

$$-\frac{x^2}{18} + \frac{6x}{9}\Big|_2^3 = \frac{7}{18}$$

(d) $E(X) = \int_0^3 x\, f(x)\, dx$

$$\int_0^3 x\left(-\frac{1}{9}x + \frac{6}{9}\right) dx$$

$$\int_0^3 -\frac{x^2}{9} + \frac{6x}{9}\, dx$$

$$-\frac{x^3}{27} + \frac{3x^2}{9}\Big|_0^3 = 2$$

4 a), b), (c)

Y \ X	1	2	3	4	5	6	$f(y)$
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{6}{12}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{6}{12}$
$f(x)$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	

d) $f(X=1 \mid Y=1) = \dfrac{\frac{1}{12}}{\frac{1}{2}} = \dfrac{1}{6}$

$\vdots$

$f(X=6 \mid Y=1) = \dfrac{\frac{1}{12}}{\frac{1}{2}} = \dfrac{1}{6}$

e) $E(X \mid Y=1) = \sum_x x\, f(x \mid Y=1)$

$= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$

$= \dfrac{21}{6}$

(f) $Var(X \mid Y=1) = \sum_x (x - E(X \mid Y=1))^2 f(x \mid Y=1)$

$= \left(1-\frac{21}{6}\right)^2\left(\frac{1}{6}\right) + \left(2-\frac{21}{6}\right)^2\left(\frac{1}{6}\right) + \dots + \left(6-\frac{21}{6}\right)^2\left(\frac{1}{6}\right)$

$= \dfrac{35}{12}$

5 $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$

$$E(\bar{X}) = E\left(\frac{1}{3}\sum_{i=1}^{3} X_i\right)$$

$$= \frac{1}{\cancel{3}}\cancel{3}\,E(X_i)$$

$$= \mu$$

$$Var(\bar{X}) = \frac{1}{9} Var(X_1 + X_2 + X_3)$$

$$= \frac{1}{9}\left(Var(X_1) + Var(X_2) + Var(X_3) + 2\,cov(X_1, X_2) + 2\,cov(X_1, X_3) + 2\,cov(X_2, X_3)\right)$$

$$= \frac{1}{9}\left(\sigma^2 + \sigma^2 + \sigma^2 + 2\left(\frac{1}{4}\sigma^2\right) + 2\left(\frac{1}{4}\sigma^2\right) + 2\left(\frac{1}{4}\sigma^2\right)\right)$$

$$= \frac{1}{9}\left(3\sigma^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2\right)$$

$$= \frac{1}{9}\left(3\sigma^2 + \frac{3}{2}\sigma^2\right)$$

$$= \frac{1}{9}\left(\frac{6\sigma^2 + 3\sigma^2}{2}\right) = \frac{1}{\cancel{9}}\,\frac{\cancel{9}\sigma^2}{2} = \frac{\sigma^2}{2}$$

6 a) $E(\bar{X}) = E\left(\frac{1}{4}\sum_{i=1}^{4} X_i\right) = \frac{1}{\cancel{4}}\cancel{4}\mu = \frac{1}{\cancel{4}}\cancel{4}E(X_i)$

$= E(X_i) = \mu$

$Var(\bar{X}) = Var\left(\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right)$

$= \frac{1}{16} Var(X_1 + X_2 + X_3 + X_4)$

$= \frac{1}{16}(Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4))$

$= \frac{1}{16}\ \sigma^2 + \sigma^2 + \sigma^2 + \sigma^2 = \frac{4\sigma^2}{16} = \frac{\sigma^2}{4}$

b) $\tilde{X} = \frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4$

$E(\tilde{X}) = E\left(\frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4\right)$

$= \frac{1}{8}E(X_1) + \frac{1}{4}E(X_2) + \frac{1}{8}E(X_3) + \frac{1}{2}E(X_4)$

$= \frac{1}{8}\mu + \frac{1}{4}\mu + \frac{1}{8}\mu + \frac{1}{2}\mu$

$= \frac{\mu + 2\mu + \mu + 4\mu}{8} = \mu.$

$\tilde{X}$ is unbiased estimator of μ.

c)

$$\begin{aligned}
var(\tilde{X}) &= var\left(\frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4\right) \\
&= var\left(\frac{X_1 + 2X_2 + X_3 + 4X_4}{8}\right) \\
&= \frac{1}{64} var\left(X_1 + 2X_2 + X_3 + 4X_4\right) \\
&= \frac{1}{64}\left(Var(X_1) + 4\,var(X_2) + var(X_3) + 16\,Var(X_4)\right) \\
&= \frac{1}{64}\left(\sigma^2 + 4\sigma^2 + \sigma^2 + 16\sigma^2\right) \\
&= \frac{22\sigma^2}{64} = \frac{11}{32}\sigma^2
\end{aligned}$$

$$Var(\bar{X}) < var(\tilde{X})$$

$\therefore$ $\bar{X}$ is better estimator for μ.