(iv) $\int \left(2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1\right) dx$

What we have to try to do is to re-arrange / or rewrite the above expression to have the forms similar to those in section 6.2 (① - ⑤)

rewrite as

$\therefore \int \left(2x^{4/5} - 7x^3 + 10e^x - 1\right) dx$

similar to ② similar to ③ similar to ①

Now view this as $\int (f(x) - g(x) + h(x) - k)\, dx$

where $f(x) = 2x^{4/5}$ $g(x) = 7x^3$ $h(x) = 10e^x$ $k = 1$

using formula ⑥ in 6.2

$$\int (2x^{4/5} - 7x^3 + 10e^x - 1)\, dx = \int f(x)dx - \int g(x)\, dx + \int h(x)\, dx - \int k\, dx \quad \text{—— (1)}$$

$$\int f(x)\, dx = \int 2 \cdot x^{4/5}\, dx$$

$\because$ 2 is a constant $\Rightarrow$ use rule ⑤ in 6.2

$$= 2\int x^{4/5}\, dx = \underbrace{2\left(\frac{x^{\frac{4}{5}+1}}{\frac{4}{5}+1} + C\right)}_{\text{use ② in 6.2, } n = \frac{4}{5}} = \frac{2x^{\frac{9}{5}}}{\frac{9}{5}} + C_1 = \frac{10}{9}x^{\frac{9}{5}} + C_1 \quad \text{—— (2)}$$

$C_1 = 2C$

$$\int g(x)\, dx = \int 7x^3\, dx = \underbrace{7\int x^3 dx}_{\text{rule ⑤}} = \underbrace{7\left(\frac{x^{3+1}}{3+1}\right)}_{\text{rule ②}} + C_2 = \frac{7}{4}x^4 + C_2 \quad \text{—— (3)}$$

$$\int h(x)\, dx = \int 10e^x\, dx = \underbrace{10\int e^x dx}_{\text{rule ⑤}} = \underbrace{10e^x + C_3}_{\text{rule ③}} \quad \text{—— (4)}$$

$$\int k\, dx = \int (1)\, dx = (1)(x) + C_4 = x + C_4 \quad \text{—— (5)}$$

Substitute (2) - (5) into (1)

$$\therefore \int (2x^{4/5} - 7x^3 + 10e^x - 1)\, dx = \frac{10}{9}x^{\frac{9}{5}} - \frac{7}{4}x^4 + 10e^x - x + C$$

$= C_1 - C_2 + C_3 - C_4$

Anyway they are just constants

$$\int \frac{(x-1)(2x+7)}{3}\,dx$$ ⇐ look here & compare to the formulas ①–⑥ in 6.2 & try to arrange this expression to look similarly to ①–⑥ in 6.2

∴ have to multiply $(x-1)$ with $(2x+7)$ to obtain a polynomial of x

$$\therefore \int \frac{(x-1)(2x+7)}{3}\,dx = \int \left(\frac{2x^2+5x-7}{3}\right)dx$$

rule 6 ⇓

$$= \int \frac{2}{3}x^2\,dx + \int \frac{5}{3}x\,dx - \int \frac{7}{3}\,dx$$

rule 5 ⇓

$$= \underbrace{\frac{2}{3}\int x^2\,dx}_{\text{rule ②}} + \underbrace{\frac{5}{3}\int x\,dx}_{\text{rule ②}} - \underbrace{\int \frac{7}{3}\,dx}_{\text{rule ①}}$$

$$* = \frac{2}{3}\frac{x^3}{3} + \frac{5}{3}\frac{x^2}{2} - \frac{7}{3}x + C$$

$$= \frac{2}{9}x^3 + \frac{5}{6}x^2 - \frac{7}{3}x + C$$

$$= \frac{1}{3}\left[\frac{2}{3}x^3 + \frac{5}{2}x^2 - 7x\right] + C$$

* Note that I did not add constants when integrating each parts but instead I add only one constant C in the end.

I can do this because either having $C_1 + C_2 + C_3$ or having $+C$ they are the same as they are all unknown constants

e.g. x^2+2x+C_1 can be seen equal to $x^2+2x+C_4+C_2+C_3$ if $C_4+C_2+C_3 = C_1$

// total cost

Remember marginal cost function = a derivative of a cost function

$$MC = C'(Q)$$

This question, MC is given and we are asked to find $C(Q)$

$\therefore$ To find $C(Q)$ is to integrate MC

$$\therefore C(Q) = \int (a + bQ + dQ^2) dQ$$

$$= \int a\,dQ + b\int Q\,dQ + d\int Q^2\,dQ$$

$$C(Q) = aQ + \frac{bQ^2}{2} + \frac{dQ^3}{3} + e$$

where e is a constant of integration

Consider $C(Q)$ and put $Q = 0$ into the expression, we will find

$$C(0) = e$$

Hence, e is in this case a fixed cost of production which is independent of Q

Given $MR = 2000 - 20Q - 3Q^2$

$\therefore$ we can find $R(Q) = \int 2000 - 20Q - 3Q^2 \, dQ$

$$= 2000Q - \frac{20Q^2}{2} - \frac{3Q^3}{3} + C$$

$$= 2000Q - 10Q^2 - Q^3 + C$$

Hint given that when $Q=0$, a revenue should be zero!! ureful hint

$$\therefore \quad R(0) = 0 = C$$

$$\therefore \quad R(Q) = 2000Q - 10Q^2 - Q^3$$

As the question asks for a demand function $D(Q)$

We should get it by now that $R(Q) = D(Q) \cdot Q$

$$\therefore \quad D(Q) = \frac{R(Q)}{Q}$$

$$= 2000 - 10Q - Q^2$$

$\#$

Show that $\int a^x\,dx = \frac{1}{\ln a} a^x + C$

This does not look like anything in ①-⑥ in 6.2

But remember in Chapter 5, a useful substitution for a is $a = e^{\ln a}$

$$\therefore \int a^x\,dx = \int \left(e^{\ln a}\right)^x dx = \int e^{(\ln a)x}\,dx$$

Now we see that we can substitute $(\ln a)\cdot x$ with u

$$\therefore \int a^x\,dx = \int e^u\,dx$$

Since $u = (\ln a)\cdot x$

$du = (\ln a)\cdot dx$ (Note $\ln a$ is only a constant)

$$\therefore \int a^x\,dx = \int \frac{e^u}{\ln a}\,du$$

$$= \frac{1}{\ln a}\int e^u\,du$$

$$= \frac{1}{\ln a} e^u + C$$

$$= \frac{1}{\ln a} e^{(\ln a)x} + C$$

$$= \frac{1}{\ln a} \left(e^{\ln a}\right)^x + C$$

$$= \frac{1}{\ln a} a^x + C$$

\#

$$\int (2t+1)e^{t^2+t}\,dt = \int (2t+1)e^u\,dt \quad \text{—(1)}$$

$$u = t^2+t$$

$$du = (2t+1)\,dt$$

$$dt = \frac{du}{2t+1}$$

$\therefore$ (1) becomes $\displaystyle\int \frac{(2t+1)e^u}{2t+1}\,du$

$$= \int e^u\,du$$

$$= e^u + C$$

$$= e^{t^2+t} + C$$

Ch6, Ex. 14

$\displaystyle\int \frac{7t}{5t^2-6}\,dt \longrightarrow$ contains fraction, this should remind you of $\int \frac{1}{u}\,du = \ln|u| + C$

However, there is $7t$ which is not a constant. :(

Luckily if you try to differentiate $5t^2-6$ with respect to t you will get $10t$ which contains t like $7t$. This is good news !!

So try substitute $u = 5t^2-6$ $\quad du = 10t\,dt \rightarrow dt = \frac{1}{10t}\,du$

$$\therefore \int \frac{7t}{5t^2-6}\,dt = \underbrace{\int \frac{7\cancel{t}}{u}\cdot\frac{1}{10\cancel{t}}\,du}_{} = \int \frac{7}{10u}\,du = \frac{7}{10}\int \frac{1}{u}\,du$$

This is why I mention about good news earlier!

$$= \frac{7}{10}\ln|u| + C$$

$$= \frac{7}{10}\ln|5t^2-6| + C$$

So I hope that you notice by now that to use substitution is to

1. Substitute such that the expression resembles to something you have integration formulas
2. If the substitution (e.g. u) are does not get rid all of t or x, don't panic. You can try anyway as you might get lucky like in this example. Otherwise you can always convert t or x in terms of u

1. $\int (3x+4)^{5/2}\,dx$

Clearly the substitution used should be $u = 3x+4$ as $\int u^{5/2}\,dx$ which looks close to $\int u^{5/2}\,du$. Moreover when trying to replace dx with $\frac{du}{3}$, this will not result in more variables e.g. the integral becomes $\int \frac{u^{5/2}}{3}\,du$

which can be integrated easily

2. $\int \frac{4}{3-x}\,dx$ only a constant

$\therefore$ can substitute $3-x$ with u to get $\int \frac{4}{u}\,dx$

Again with this substitution $dx = -du$ ($\because u = 3-x$) which does not contain x variables so the integral becomes $\int -\frac{4}{u}\,du$ which can be integrated

3. $\int t e^{2-t^2}\,dt$

For this one we might get worried if substitute $2-t^2$ with u because the integral still contain a variable $t \Rightarrow \int t e^u\,dt$

But you won't get too worried if you notice that $du = -2t\,dt$ which also contains t multiply with a constant (-2, in this case).

This is because when you try to substitute and change dt to something in terms of du, the integral becomes

$$\int \frac{t e^u}{-2t}\,du = \int -\frac{1}{2} e^u\,du = -\frac{1}{2}\int e^u\,du$$

This can be integrated easily.

4. $\int t(2+t^2)^3\,dt$

$\hookrightarrow u$

$du = 2t\,dt$

$\hookrightarrow$ containing t like in the integral

Same order of power of variable t

$\therefore$ the substitution of $u = 2+t^2$ is o.k.

(you can check this by trying to substitute $(2+t^2)$ with u and write $dt = \frac{du}{2t}$)

5. substitution used should be $(2x-5) = u$

The integral becomes $\int \frac{3}{u^4}\,dx$ and if you try to write dx in terms of du, it won't create more variables x as $dx = \frac{du}{2}$. $\therefore$ The integral becomes $\int \frac{3}{u^4}\cdot\frac{1}{2}\,du = \frac{3}{2}\int\frac{1}{u^4}\,du$

$= \frac{3}{2}\int u^{-4}\,du$

Now, you can integrate this integral

6. $\int x^2 e^{-x^3}$ $\rightarrow$ diff this results in something of x^2

the same order of power over x variables

$\therefore$ The substitution $u = -x^3$ or x^3 is OK.

7. $\int \frac{e^t}{e^t+1}$ $\rightarrow$ diff this results in e^t, which is the same as here

$\therefore$ The substitution $u = e^t+1$ is fine

8. $u = t^2+6t+5$ $\quad du = (2t+6)\,dt$

$\hookrightarrow$ $2(t+3)$ and $t+3$ presents in the integral so integral becomes $\int \frac{\cancel{t+3}}{u^{1/3}}\cdot\frac{1}{2\cancel{(t+3)}}\,du = \int u^{-1/3}\,du$

$\int \left(\frac{x^3+x^2-x-3}{x^2-3} \right) dx$ again substitution of $u = x^2-3$ won't help because we still have x^3+x^2-x-3 on the top of the fraction. (Note $du = 2x\,dx$ containing x to power 1 only)

As we notice that $x^3+x^2-x-3 > x^2-3$, we can actually try to divide x^3+x^2-x-3 with x^2-3

$$\begin{array}{r|l} & x+1 \\ \hline x^2-3 & x^3+x^2-x-3 \\ & x^3 \quad -3x \\ \hline & x^2+2x-3 \\ & x^2 \quad -3 \\ \hline & 2x \end{array}$$

$2x$ cannot divide further

$\therefore \frac{x^3+x^2-x-3}{x^2-3} = x+1+\frac{2x}{x^2-3}$

$\int \left(x+1+\frac{2x}{x^2-3} \right) dx = \int x\,dx + \int 1\,dx + \int \frac{2x}{u}\,dx$

$u = x^2-3 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$\therefore \int x\,dx + \int 1\,dx + \int \frac{\cancel{2x}}{u} \frac{1}{\cancel{2x}} du = \frac{x^2}{2} + x + \ln|x^2-3| + C$

Ch6, Ex.17 $\quad \frac{9x^2+5}{3x} = 3x + \frac{5}{3x}$

$\int \frac{9x^2+5}{3x} dx = \int 3x\,dx + \int \frac{5}{3x} dx$

$= \frac{3x^2}{2} + \frac{5}{3} \ln|x| + C$

Ch6, Ex.18

$$\begin{array}{r|l} & 2x-3 \\ \hline 3x-1 & 6x^2-11x+5 \\ & 6x^2-2x \\ \hline & -9x+5 \\ & -9x+3 \\ \hline & 2 \end{array}$$

$2 \rightarrow$ cannot be divided further

$\therefore \int 2x\,dx - \int 3\,dx + \int \frac{2}{3x-1} dx = \frac{2x^2}{2} - 3x + 2\int \frac{1}{u} dx$

$\therefore u = 3x-1 \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3} du \quad \therefore \int = x^2-3x+\frac{2}{3}\ln|u| + C$

$= x^2-3x+\frac{2}{3}\ln|3x-1| + C$

Ch6, Ex. 21.

$$y = \int (3-2x)^2\,dx$$

$$= \int u^2\,dx \quad \text{where } u = 3-2x \Rightarrow du = -2dx \Rightarrow dx = -\frac{1}{2}du$$

$$= \int u^2\left(-\frac{1}{2}\right)du$$

$$= -\frac{1}{2}\int u^2\,du$$

$$y = -\frac{1}{2}\frac{u^3}{3} + C = \frac{-u^3}{6} + C = -\frac{1}{6}(3-2x)^3 + C$$

Given $y(0)=1$ $\therefore$ $1 = -\frac{1}{6}(3)^3 + C = -\frac{9}{2}+C$

$$C = \frac{11}{2}$$

$$\therefore \quad y = -\frac{1}{6}(3-2x)^3 + \frac{11}{2} \quad \times$$

Ch6, Ex.22 $\quad y' = \int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C_1$

$$y' = \frac{-1}{x} + C_1$$

Given $y'(-1) = 1$

$$\therefore \quad 1 = \frac{-1}{-1} + C_1 \Rightarrow C_1 = 0$$

$$\therefore y' = \frac{-1}{x}$$

$$y = \int -\frac{1}{x}\,dx$$

Note $\int \frac{1}{x}\,dx = \ln|x|$

not $\frac{x^{-1+1}}{-1+1}$

$$= -\ln|x| + C_2$$

Given $y(1) = 0$

$$\therefore \quad 0 = \underbrace{-\ln(1)}_{=0} + C_2$$

$$\therefore C_2 = 0$$

$$\therefore y = -\ln|x| \quad \times$$

(ii) $y = 2 - x^2, \quad y = x$

① Find intersection $\quad 2 - x^2 = x$

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$$x = 1, -2$$

② Justify positive, negative areas

As the limits for integration are 1 & -2 which are the same as intersection points, we do not need to split the integrals.

If the answer is negative, we can simply multiply with -1 to give the answer in terms of area.

We can also foresee whether we are going to get number ⊖ or ⊕ by i) sketch graph (I'm going to skip this)

ii) choose a value of x between 1 & -2 e.g. 0

If the area is found by $\int \underbrace{2 - x^2 - x}_{g(x)} \, dx$

$$g(0) = 2 - 0 - 0 = 2 = +$$

meaning that we would get positive number as an answer.

Let's see $\Rightarrow \displaystyle\int_{-2}^{1} 2 - x^2 - x \, dx = \left[2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{-2}^{1}$

$$= 2(1) - \frac{1}{3} - \frac{1}{2} - \left(-4 + \frac{8}{3} - 2 \right)$$

$$= 2 - \frac{1}{3} - \frac{1}{2} + 4 - \frac{8}{3} + 2$$

$$= 8 - \frac{1}{3} - \frac{8}{3} - \frac{1}{2} = 8 - \frac{9}{3} - \frac{1}{2} = 4\frac{1}{2} = \frac{9}{2}$$

Ch6, Ex 32

Try $u = 1+\sqrt{x} = 1+x^{1/2}$

$du = \frac{1}{2}x^{-1/2}dx \Rightarrow dx = 2\sqrt{x}\,du$

$$\int_9^{64}\left(\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\right)dx = \int \frac{u^{1/2}}{\cancel{\sqrt{x}}}\,2\cancel{\sqrt{x}}\,du$$

cancel out $\Rightarrow$ lucky!! :)

Need to change limits to values of u as well !!

$x=9 \Rightarrow u = 1+\sqrt{9} = 4$, $x = 64 \Rightarrow u = 1+\sqrt{64} = 9$

$$\therefore \int_9^{64}\left(\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\right)dx = 2\int_4^9 u^{1/2}\,du$$

$$= \left[\frac{2u^{3/2}}{\frac{3}{2}}\right]_4^9 = \frac{4}{3}\left[u^{3/2}\right]_4^9$$

$$= \frac{4}{3}(27-8) = \frac{4\times19}{3} = \frac{76}{3}$$ ✗