EE320 (2/2012)

INTRODUCTORY MATHEMATICAL ECONOMICS

DERIVATIVES OF MORE-THAN-ONE INDEPENDENT VARIABLE FUNCTION

(Part 1)

Topics

- Partial Differentiation
  - First-order partial derivatives
  - Second-order partial derivatives
- Differentials
- Total differentials
- Total derivatives
- Implicit function and its derivative

Partial Differentiation

- Consider $y = f(x_1, x_2, ..., x_n)$ where $x_1, ..., x_n$ are independent of one another.
- Suppose only x_1 changes by Δx_1, the corresponding change in y is Δy.
- Difference quotient:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, ..., x_n) - f(x_1, x_2, ..., x_n)}{\Delta x_1}$$

- Partial derivative of y with respect to x:

- The process of taking partial derivatives is called "partial differentiation".

First-Order Partial Derivatives

- Technique of partial differentiation: hold (n-1) independent variables constant while allowing one variable to vary.
- Example 1: $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$. Find $f_1(1,3)$ and $f_2(1,3)$.
- Example 2: $y = f(u,v) = (u+4)(3u+2v)$. Find $f_u(2,1)$ and $f_v(2,1)$.

Geometric Interpretation of Partial Derivatives

- Suppose Q = Q(K, L)
- $\partial Q/\partial L =$

Gradient Vector

- The gradient vector of the function $f(x_1, x_2, \ldots, x_n)$ is an n-vector of all the partial derivatives:

$$\nabla f(x_1, x_2, \ldots, x_n) = (f_1, f_2, \ldots, f_n)$$

where $f_i \equiv \frac{\partial y}{\partial x_i}$.

- Example: Find the gradient vector of the production function $Q = aK^b L^{1-b}$, where $a > 0$ and $0 < b < 1$.

Second-Order Partial Derivatives (1)

- Consider the function *z = f(x, y),* which give rise to:

$$f_x \equiv \frac{\partial z}{\partial x} \quad \text{and} \quad f_y \equiv \frac{\partial z}{\partial y}$$

- Since f_x is a function of *x* (and *y*), we can determine the rate of change of f_x with respect to *x*, while *y* is fixed, by a *second-order* partial derivative with respect to *x*:

or

- Similarly, the *second-order* partial derivative with respect to *y* is:

or

Second-Order Partial Derivatives (2)

- Also, since f_x is a function of *y* and f_y is a function of *x*, *cross (or mixed) partial derivatives* can be written as:

and

- **Young's Theorem**:

Let $y = f(x_1, x_2, \ldots, x_n)$ is twice continuously differentiable (C^2). Then,

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_j \partial x_i} ; i \neq j$$

Second-Order Partial Derivatives (3)

- Let $y = f(x_1, x_2)$. The second-order partial derivatives can be written in a matrix form called "Hessian matrix":

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

- Example: Let $z = x^2 e^{-y}$. Find f_x, f_y, and the Hessian matrix.

Differentials

- Recall the definition of derivatives:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

➔ $dy = f'(x)dx$

  - dy = differential of y
  - dx = differential of x

- Example: Let $y = 3x^2 + 7x - 5$. Find dy.

- The process of finding the differential *dy* from a function $y = f(x)$ is called *differentiation*. (Note: for finding *dy/dx*, this is called differentiation with respect to x).

Total Differentials (1)

- Consider a saving function $S = S(Y, r)$ where Y = income, i = interest rate.
  - Marginal propensity to save = $\frac{\partial S}{\partial Y}$
  - For a *given change in Y (dY)*, the resulting change in S can be approximated by: $dS = \left(\frac{\partial S}{\partial Y}\right) \cdot dY$
  - For a *given change in r (dr)*, the resulting change in S can be approximated by: $dS = \left(\frac{\partial S}{\partial r}\right) \cdot dr$
  - The total change in S is approximated by the total differential:

or

Total Differentials (2)

- Total differential of a function Y is the sum of the approximated changes from all parameters.
- The process of finding total differentials is called "total differentiation."
- Suppose r is constant. Then, dr = 0.

➔ $dS = S_Y dY + S_r dr$

➔

i.e. partial derivative (∂S/∂Y) is the ratio of two differentials (dS/dY) when r is held constant.

Total Differentials (3)

- General case of *n* independent variables:

$$U = U(x_1, x_2, \ldots, x_n)$$

- Total differential of U is:

$$dU = \left(\frac{\partial U}{\partial x_1}\right) \cdot dx_1 + \left(\frac{\partial U}{\partial x_2}\right) \cdot dx_2 + \ldots + \left(\frac{\partial U}{\partial x_n}\right) \cdot dx_n$$

or

$$dU = U_1 dx_1 + U_2 dx_2 + \ldots + U_n dx_n = \sum_{i=1}^{n} U_i dx_i$$

Examples: Total Differentials

- Point elasticities of saving:

$$\varepsilon_{SY} = \frac{\partial S / \partial Y}{S / Y} = \frac{\partial S}{\partial Y} \cdot \frac{Y}{S} \quad \text{and} \quad \varepsilon_{Sr} = \frac{\partial S / \partial r}{S / r} = \frac{\partial S}{\partial r} \cdot \frac{r}{S}$$

- *n* Partial derivative of utility function:

$$\varepsilon_{Ux_i} = \frac{\partial U}{\partial x_i} \cdot \frac{x_i}{U}$$

  - Example 1: $U(x_1, x_2) = ax_1 + bx_2$
  - Example 2: $U(x_1, x_2) = x_1^a x_2^b$

Rules of Differentials

- Let $y = f(x_1, x_2)$

  $dy = f_1 dx_1 + f_2 dx_2$, where $f_1 = \partial y/\partial x_1$, $f_2 = \partial y/\partial x_2$

- **Rules of differentials:**

Rule I $dk = 0$

Rule II $d(cu^n) = cnu^{n-1}du$

Rule III $d(u \pm v) = du \pm dv$

Rule IV $d(u \cdot v) = v \cdot du + u \cdot dv$

Rule V $d(u/v) = [v \cdot du - u \cdot dv]/v^2$

Rule VI $d(u \pm v \pm w) = du \pm dv \pm dw$

Rule VII $d(uvw) = vw \cdot du + uw \cdot dv + uv \cdot dw$

Examples: Rules of Differentials

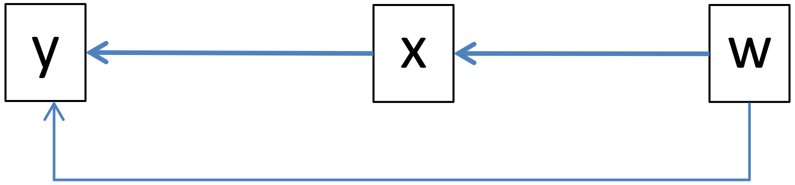
- Example 1: $y = 3x_1^2 + x_1 x_2^2$

- Example 2: $y = \dfrac{x_1 + x_2}{2x_1^2}$

- Example 3: $y = 3x_1(2x_2 - 1)(x_3 + 5)$

Total Derivatives (1)

- Now, consider the case where independent variables are *related* to one another.
  - Example: what is the rate of change of $C(Y^*, T_0)$ w.r.t. T_0, where Y and T_0 are related?
- Consider y = f(x, w) where x = g(w)



➔ y = f[g(w), w]

➔ dy/dw = ?

Total Derivatives (2)

- From $y = f[g(w), w]$

1. Use chain rule:

2. Totally differentiate:

⇨

- dy/dw is called the "total derivative of y w.r.t. x".
- ∂y/∂w (partial derivative) is a component of the total derivative.

Examples: Total Derivatives

- Example 1: $y = f(x, w) = 3x – w^2$, where $x = g(w) = 2w^2+w+4$

- Example 2: $U = U[c, g(c)]$

A Variation on Total Derivatives

- Let $y = f(x_1, x_2, w)$ where $\begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases}$

- Example: $Q = Q(K, L, t)$, where $K = K(t)$ and $L = L(t)$

Another Variation on Total Derivatives

- Let $y = f(x_1, x_2, u, v)$ where $\begin{cases} x_1 = g(u,v) \\ x_2 = h(u,v) \end{cases}$

Implicit Function

- Consider $y = f(x) = 3x^4$: Explicit function

  ➔ $y - 3x^4 = 0$: Implicit function

- General form: $F(y, x_1, x_2, \ldots, x_n) = 0$.
- This function may define an implicit function:

$$y = f(x_1, x_2, \ldots, x_n)$$

Derivatives of Implicit Functions

- Given $F(y, x_1, x_2, \ldots, x_n) = 0$,
- We can write $dF = d(0) = 0$,

  or

- Implicit function has the total differential:
- Substituting dy in (*) gives:

➔ $$f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}$$:Implicit Function Rule

Example: Derivatives of Implicit Functions

- Example 1: Find dy/dx for the implicit function $y - 3x^4 = 0$.

- Example 2: Find ∂y/∂x for any implicit function(s) that may be defined by $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$.

- Example 3: Assume $F(Q, K, L) = 0$ implicitly defines the production function $Q = f(K,L)$. Express MPP_K and MPP_L in relation to the function F.