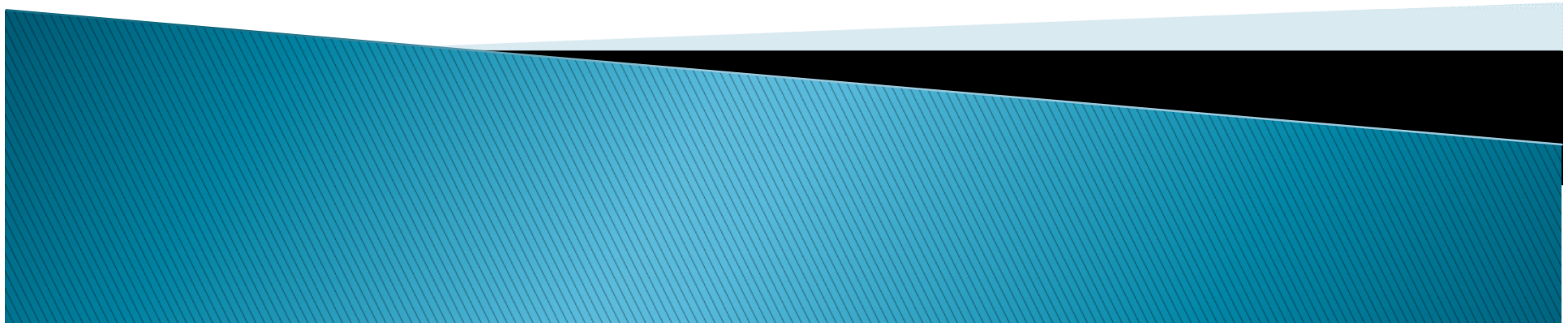


**Extensions of the Two-Variable Linear
Regression Model
Part I**



Scaling and Units of Measurement

Consider the data given in Table 6.2, which refers to U.S. gross private domestic investment (GPDI) and gross domestic product (GDP) in billions as well as millions of (chained) 2000 dollars.

Suppose in the regression of GPDI on GDP one researcher uses data in billions of dollars but another expresses data in millions of dollars.

- **Will the regression results be the same in both cases?**
- **Do the units in which the regressand and regressor are measured make any difference in the regression results?**



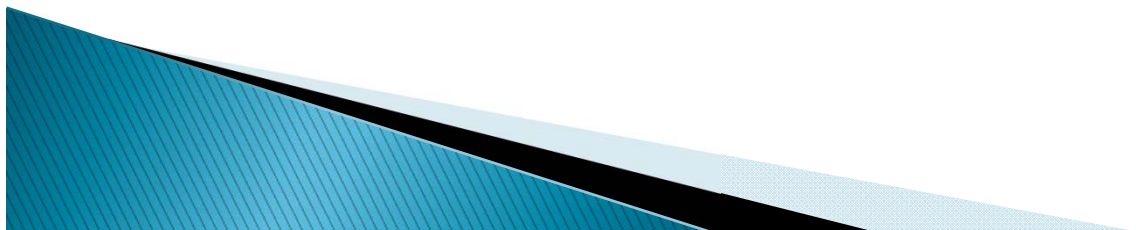
$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i$$

where $Y = \text{GDPI}$ and $X = \text{GDP}$

$$Y_i^* = w_1 Y_i$$

$$X_i^* = w_2 X_i$$

Where w_1 and w_2 are constants, call the **Scale factors**



- ▶ If Y_i and X_i are measured in billions of dollars and one wants to express them in millions of dollars, we will have

$$Y_i^* = 1000 Y_i$$

$$X_i^* = 1000 X_i$$

$$w_1 = w_2 = 1000$$



$$Y_i^* = \hat{\beta}_1^* + \hat{\beta}_2^* X_i + \hat{u}_i$$

where $Y_i^* = w_1 Y_i$, $X_i^* = w_2 X_i$, and $\hat{u}_i^* = w_1 \hat{u}_i$



$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$$

$$\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$\text{var}(\hat{\beta}_1) = \frac{\sum X_i^2}{n \sum x_i^2} \sigma^2$$

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2}$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2}$$

$$\hat{\beta}_1^* = \bar{Y}^* - \hat{\beta}_2^* \bar{X}^*$$

$$\hat{\beta}_2^* = \frac{\sum x_i^* y_i^*}{\sum x_i^{*2}}$$

$$\text{var}(\hat{\beta}_1^*) = \frac{\sum X_i^{*2}}{n \sum x_i^{*2}} \sigma^{*2}$$

$$\text{var}(\hat{\beta}_2^*) = \frac{\sigma^{*2}}{\sum x_i^{*2}}$$

$$\hat{\sigma}^{*2} = \frac{\sum \hat{u}_i^{*2}}{n-2}$$



$$\hat{\beta}_2^* = \left(\frac{w_1}{w_2} \right) \hat{\beta}_2$$

$$\hat{\beta}_1^* = w_1 \hat{\beta}_1$$

$$\hat{\sigma}^{*2} = w_1^2 \hat{\sigma}^2$$

$$\text{var}(\hat{\beta}_1^*) = w_1^2 \text{var}(\hat{\beta}_1)$$

$$\text{var}(\hat{\beta}_2^*) = \left(\frac{w_1}{w_2} \right)^2 \text{var}(\hat{\beta}_2)$$

$$r_{xy}^2 = r_{x^*y^*}^2$$



Example

Gross Private Domestic Investment and GDP, United States, 1990-2005

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i$$

where $Y_i = \text{GPDI}$ and $X_i = \text{GDP}$

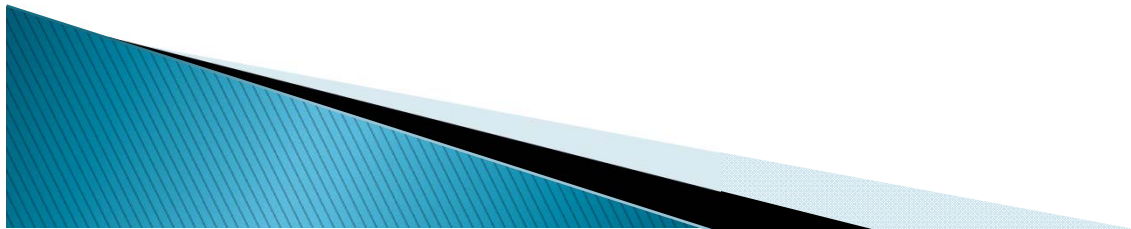


TABLE 6.2
Gross Private Domestic Investment and GDP, United States, 1990–2005
 (Billions of chained [2000] dollars, except as noted; quarterly data at seasonally adjusted annual rates)

Source: *Economic Report of the President, 2007*, Table B-2, p. 328.

Year	GPDIBL	GPDIM	GDPB	GDPM
1990	886.6	886,600.0	7,112.5	7,112,500.0
1991	829.1	829,100.0	7,100.5	7,100,500.0
1992	878.3	878,300.0	7,336.6	7,336,600.0
1993	953.5	953,500.0	7,532.7	7,532,700.0
1994	1,042.3	1,042,300.0	7,835.5	7,835,500.0
1995	1,109.6	1,109,600.0	8,031.7	8,031,700.0
1996	1,209.2	1,209,200.0	8,328.9	8,328,900.0
1997	1,320.6	1,320,600.0	8,703.5	8,703,500.0
1998	1,455.0	1,455,000.0	9,066.9	9,066,900.0
1999	1,576.3	1,576,300.0	9,470.3	9,470,300.0
2000	1,679.0	1,679,000.0	9,817.0	9,817,000.0
2001	1,629.4	1,629,400.0	9,890.7	9,890,700.0
2002	1,544.6	1,544,600.0	10,048.8	10,048,800.0
2003	1,596.9	1,596,900.0	10,301.0	10,301,000.0
2004	1,713.9	1,713,900.0	10,703.5	10,703,500.0
2005	1,842.0	1,842,000.0	11,048.6	11,048,600.0

Note: GPDIBL = gross private domestic investment, billions of 2000 dollars.
 GPDIM = gross private domestic investments, millions of 2000 dollars.
 GDPB = gross domestic product, billions of 2000 dollars.
 GDPM = gross domestic product, millions of 2000 dollars.

GPDIBL = Gross private domestic investment, billions of 2000 dollars

GPDIM = Gross private domestic investment, millions of 2000 dollars

GDPB = Gross domestic product, billions of 2000 dollars

GDPM = Gross domestic product, millions of 2000 dollars

Both GPDI and GDP in billions of dollars:

$$\widehat{GPDI}_t = -926.090 + 0.2535GDP_t$$
$$se = (116.358) \quad (0.0129)$$
$$r^2 = 0.9648$$

GPDI in billions of dollars → millions of dollars

GDP in billions of dollars → millions of dollars



$$w_1 = 1000$$

$$w_2 = 1000$$

$$w_1 \hat{\beta}_1 = 1000 * -926.090 = -926,090$$

$$\frac{w_1}{w_2} \hat{\beta}_2 = \frac{1000}{1000} * 0.2535 = 0.2535$$

$$\widehat{GPD\bar{I}}_t = -926,090 + 0.2535GDP_t$$

$$se = (116,358) \quad (0.0129)$$

$$r^2 = 0.9648$$



Both GPDI and GDP in millions of dollars:

$$\widehat{GPDI}_t = -926,090 + 0.2535GDP_t$$
$$se = (116,358) \quad (0.0129)$$
$$r^2 = 0.9648$$

GPDI in millions of dollars \longrightarrow billions of dollars

GDP in millions of dollars \longrightarrow billions of dollars



$$w_1 = \frac{1}{1000}$$

$$w_2 = \frac{1}{1000}$$

$$w_1 \hat{\beta}_1 = \frac{1}{1000} * -926,090 = -926.090$$

$$\frac{w_1}{w_2} \hat{\beta}_2 = \frac{\frac{1}{1000}}{\frac{1}{1000}} * 0.2535 = 0.2535$$

$$\widehat{GPD}I_t = -926.090 + 0.2535GDP_t$$

$$se = (116.358) \quad (0.0129)$$

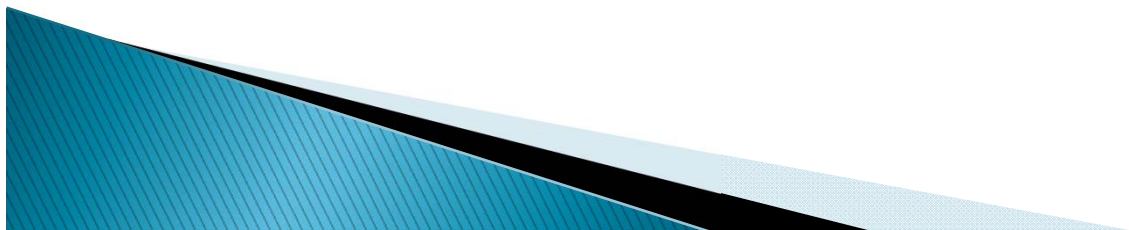
$$r^2 = 0.9648$$

Both GDPI and GDP in billions of dollars:

$$\begin{aligned}\widehat{GDPI}_t &= -926.090 + 0.2535GDP_t \\ se &= (116.358) \quad (0.0129) \\ r^2 &= 0.9648\end{aligned}$$

GDPI in billions of dollars \longrightarrow billions of dollars

GDP in billions of dollars \longrightarrow millions of dollars



$$w_1 = 1$$
$$w_2 = 1000$$

$$w_1 \hat{\beta}_1 = 1 * -926.090 = -926.090$$

$$\frac{w_1}{w_2} \hat{\beta}_2 = \frac{1}{1000} * 0.2535 = 0.0002535$$

$$\widehat{GPD\hat{I}}_t = -926.090 + 0.0002535GDP_t$$
$$se = (116.358) \quad (0.0000129)$$
$$r^2 = 0.9648$$

Both GPDI and GDP in millions of dollars:

$$\widehat{GPDI}_t = -926,090 + 0.2535GDP_t$$
$$se = (116,358) \quad (0.0129)$$
$$r^2 = 0.9648$$

GDPI in millions of dollars \longrightarrow billions of dollars

GDP in millions of dollars \longrightarrow millions of dollars



$$w_1 = \frac{1}{1000}$$
$$w_2 = 1$$

$$w_1 \hat{\beta}_1 = \frac{1}{1000} * -926,090 = -926.090$$

$$\frac{w_1}{w_2} \hat{\beta}_2 = \frac{1}{1000} * 0.2535 = 0.0002535$$

$$\widehat{GPD\bar{I}}_t = -926.090 + 0.0002535 GDP_t$$

$$se = (116.358) \quad (0.0000129)$$

$$r^2 = 0.9648$$

Both GDPI and GDP in billions of dollars:

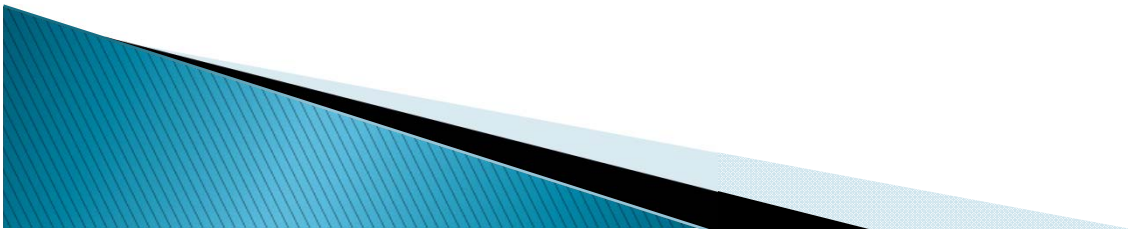
$$\boxed{GDPI}_t = -926.090 + 0.2535GDP_t$$

$$se = (116.358) \quad (0.0129)$$

$$r^2 = 0.9648$$

GDPI in billions of dollars \longrightarrow millions of dollars

GDP in billions of dollars \longrightarrow billions of dollars



$$w_1 = 1000$$

$$w_2 = 1$$

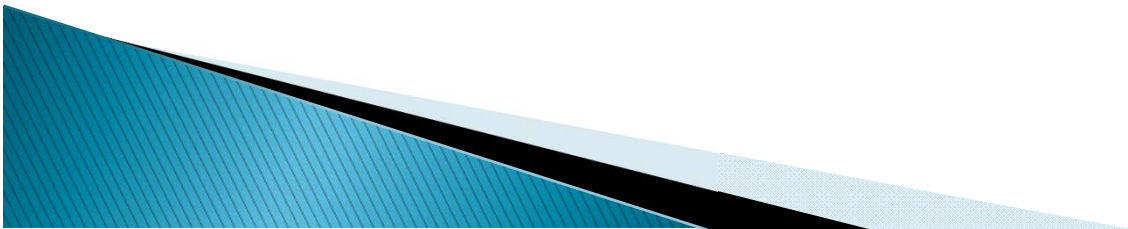
$$\hat{\beta}_1^* = w_1 \hat{\beta}_1 = 1000 * -926.090 = -926,090$$

$$\hat{\beta}_2^* = \left(\frac{w_1}{w_2} \right) \hat{\beta}_2 = \left(\frac{1000}{1} \right) 0.2535 = 253.524$$

$$\square \text{G}PDI_t = -926,090 + 253.524 \text{G}DP_t$$

$$se = (116,358) \quad (12.9465)$$

$$r^2 = 0.9648$$



Both GDPDI and GDP in millions of dollars:

$$\begin{aligned}\widehat{GDPDI}_t &= -926.090 + 0.2535GDP_t \\ se &= (116.358) \quad (0.0129) \\ r^2 &= 0.9648\end{aligned}$$

GDPDI in millions of dollars \longrightarrow millions of dollars

GDP in millions of dollars \longrightarrow billions of dollars



$$w_1 = 1$$

$$w_2 = \frac{1}{1000}$$

$$w_1 \hat{\beta}_1 = 1 * -926,090 = -926,090$$

$$\frac{w_1}{w_2} \hat{\beta}_2 = \frac{1}{\frac{1}{1000}} * 0.2535 = 253.524$$

$$\widehat{GPDI}_t = -926,090 + 253.524 GDP_t$$

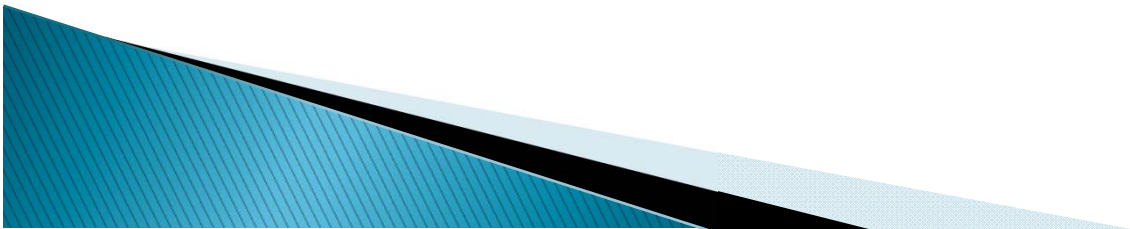
$$se = (116,358) \quad (12.9465)$$

$$r^2 = 0.9648$$



Regression through the origin

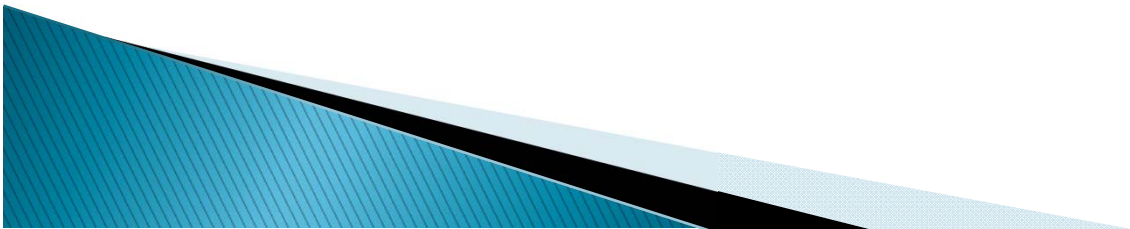
$$Y_i = \beta_2 X_i + u_i$$



Regression through the origin

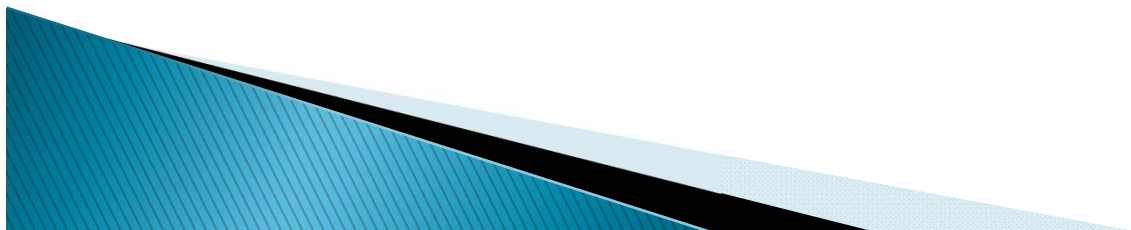
$$\hat{\beta}_2 = \frac{\sum X_i Y_i}{\sum X_i^2}$$

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum X_i^2}, \text{ where } \sigma^2 = \frac{\sum \hat{u}_i^2}{n-1}$$



R-squared for Regression through Origin Model

$$\text{raw } r^2 = \frac{(\sum X_i Y_i)^2}{\sum X_i^2 \sum Y_i^2}$$



The capital asset pricing model (CAPM)

- ▶ Risk premium form may be expressed as

$$(ER_i - r_f) = \beta_i (ER_m - r_f)$$

Where

ER_i = expected rate of return on security i

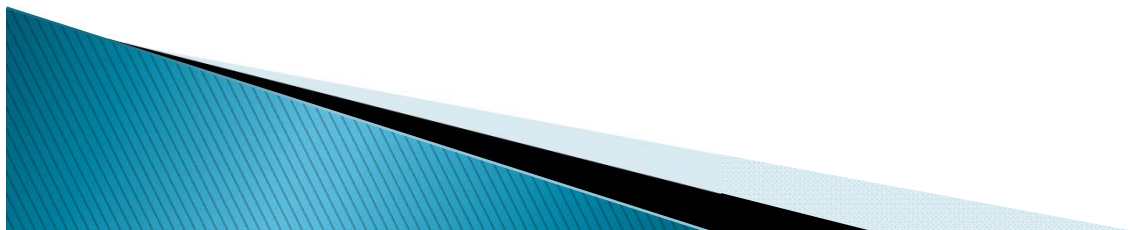
ER_m = expected rate of return on the market portfolio as represented by, say, the S&P 500 companies stock index

r_f = risk-free rate of return, say, the return on 90 day Treasury bills

β_i = the Beta coefficient, a measure of systematic risk, risk that cannot be eliminated through diversification. A measure of the extent to which the i th security rate of return moves with the market. A $\beta_i > 1$ implies a volatile or aggressive security, whereas $\beta_i < 1$ suggests a defensive security



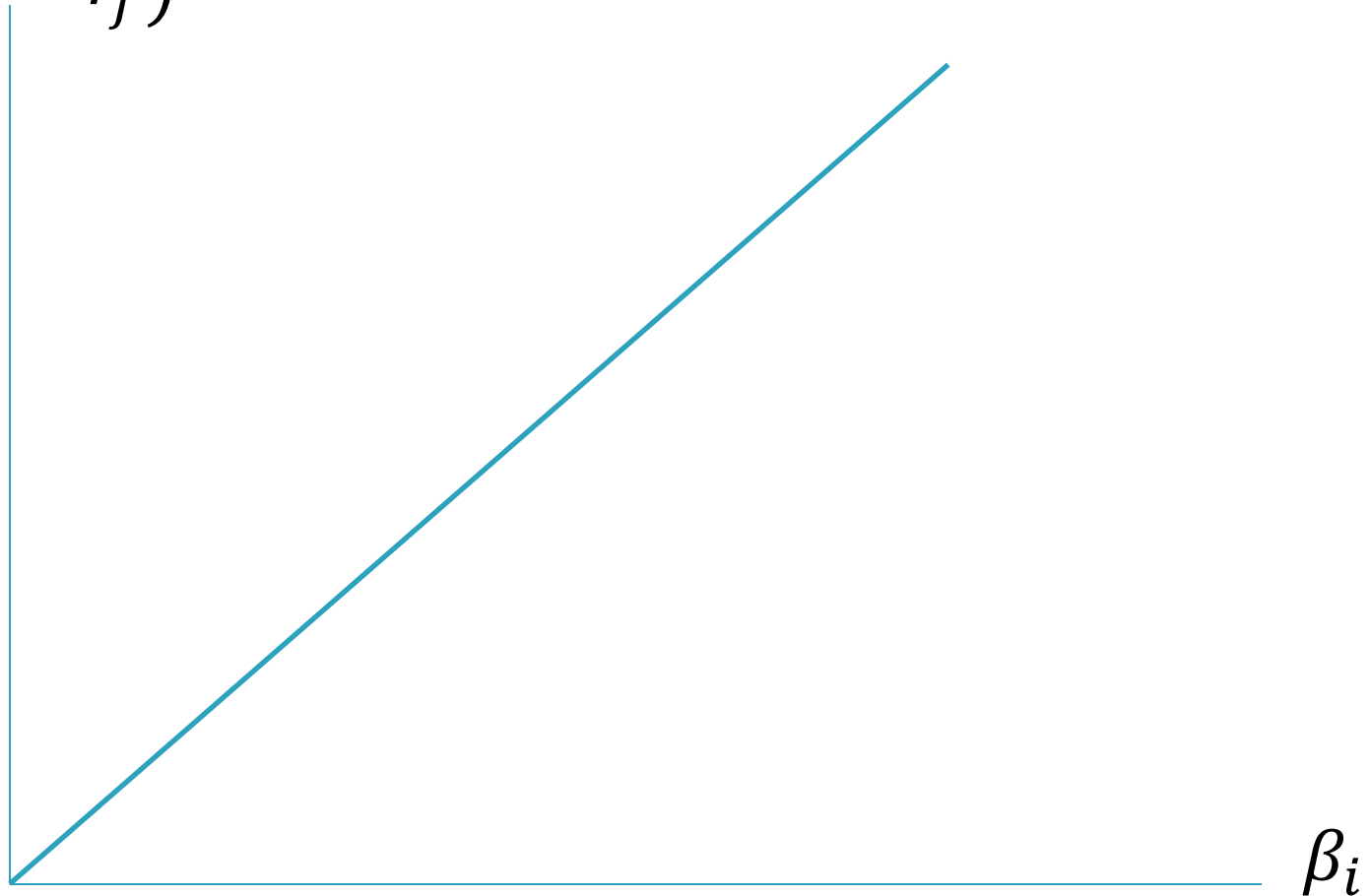
- ▶ If capital market works efficiently, then CAPM postulates that security i 's expected risk premium ($ER_i - r_f$) is equal to that security's β coefficient times the expected market risk premium ($ER_m - r_f$)
- ▶ The line shown in the figure is known as the security market line (SML)



Systematic risk

$$(ER_i - r_f)$$

security market line



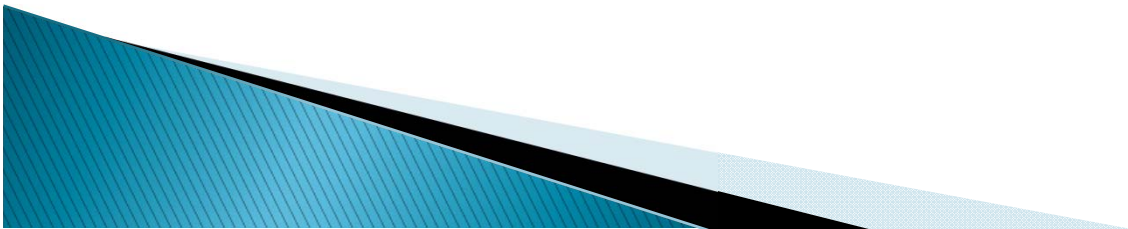
- ▶ For empirical purposes, the equation is often expressed as

$$(R_i - r_f) = \beta_i (R_m - r_f) + u_i$$

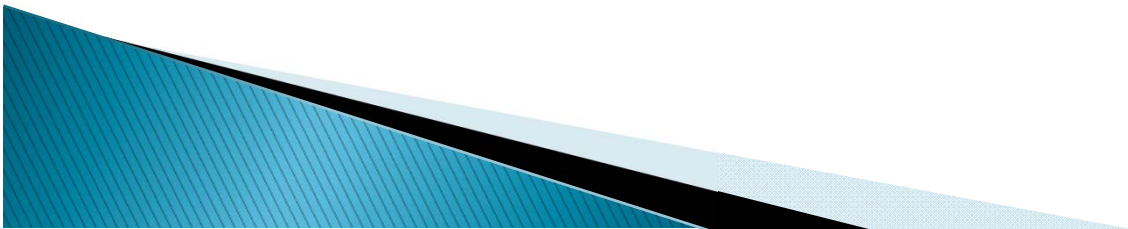
Or

$$(R_i - r_f) = \alpha_i + \beta_i (R_m - r_f) + u_i$$

The latter model is known as the Market Model. If CAPM holds, α_i is expected to be zero.



- ▶ The dependent variable, Y is $(R_i - r_f)$
- ▶ The explanatory variable X is β_i the volatile coefficient

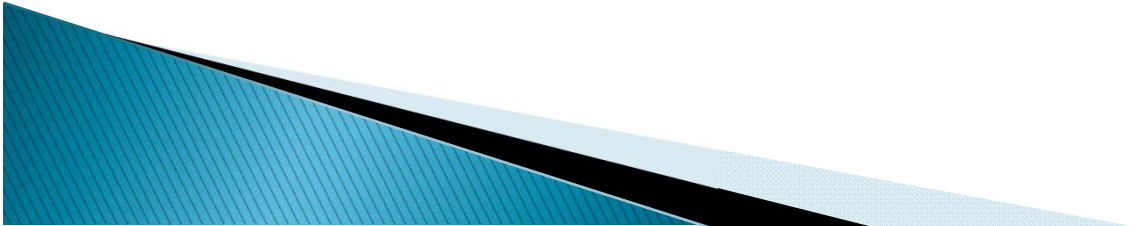
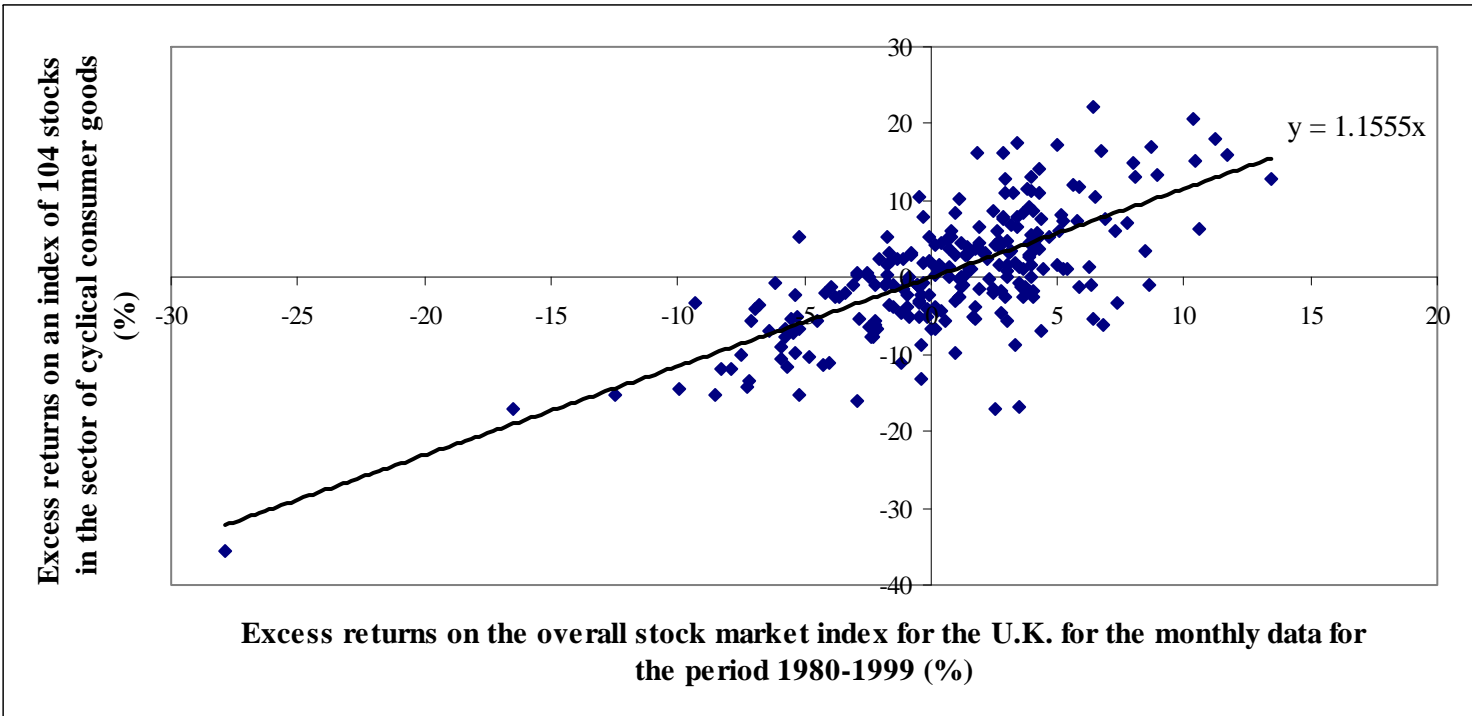


Example

Table 6.1 (P.151) gives data on excess returns Y_t (%) on an index on 104 stocks in the sector of cyclical consumer goods and excess returns X_t (%) on the overall stock market index for the U.K. for the monthly data for the period 1980-1999, for a total of 240 observations.

Excess returns refers to return in excess of return on a riskless asset. (Capital asset pricing model, CAPM)





Dependent variable: Y

Method: OLS

Sample: 1980 M01 1999 M12

N = 240

	Coefficient	Std. Error	T-stat	p-value
X	1.155512	0.074396	15.5320	0.0000

$$R^2 = 0.500309$$

	Coefficient	Std. Error	T-stat	p-value
X	1.171128	0.075386	15.5350	0.0000
Constant	-0.447481	0.362943	-1.232924	0.2188

$$R^2 = 0.503480$$


The slope coefficient is **highly significant**

If the excess market rate goes up by **1 percentage point**, the excess return on the index of consumer goods sector goes up by about **1.15 percentage points**.

If a Beta coefficient is greater than 1, such a security is said to be volatile; it moves more than proportionately with the overall stock market index



Source

Gujarati, D.N. (2009) Basic Econometrics. 5th ed.
Singapore, McGraw-Hill. (G)

