EE320 (2/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

NONLINEAR MODEL AND DIFFERENTIAL CALCULUS
IN ECONOMIC THEORY

Topics

- Slopes of Curves: Linear and Non-linear Models
- Slope and Derivatives of a Function
- Differentiability of a Function
- Rules of Differentiation
- Examples in Economics:
  - Derivative and Marginality
  - Relations among the total, the average, and the marginal functions

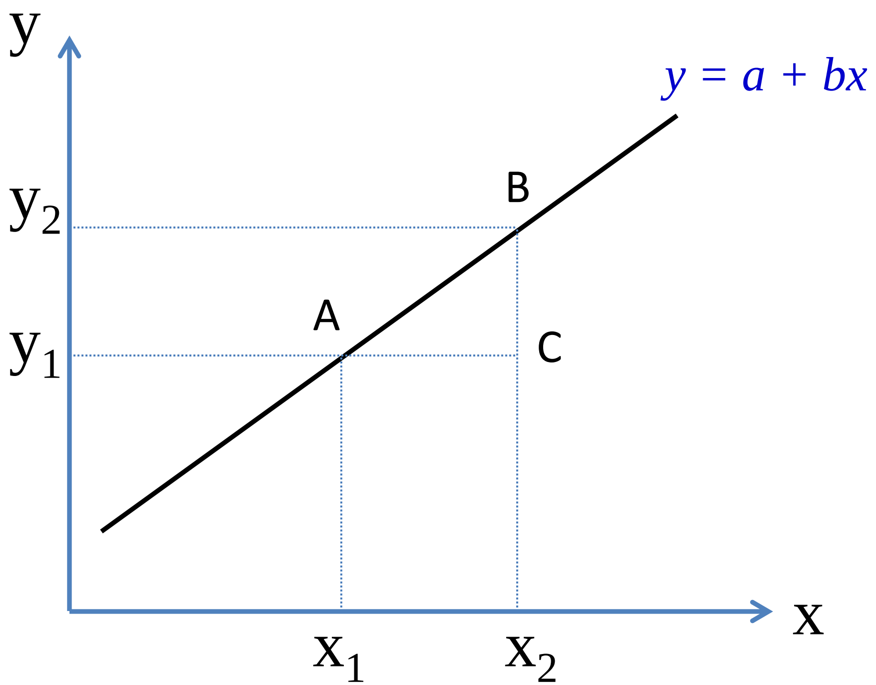
REVIEW DIFFERENTIAL CALCULUS

Introduction

- This lecture focuses the *rate of change* of the equilibrium value of an endogenous variable (*y*) with respect to the change in a particular exogenous variable (*x*), given $y = f(x)$.
- Geometrically, this rate of change is referred to as the slope of a curve.
- Examples:
  - Slope of a total cost function is the marginal cost.
  - Slope of a utility function is marginal utility.
- The slope of a function tells the characteristics of the function.
  - Linear functions: Slope is constant.
  - Nonlinear functions: Slopes vary at different values of x.

Linear Function

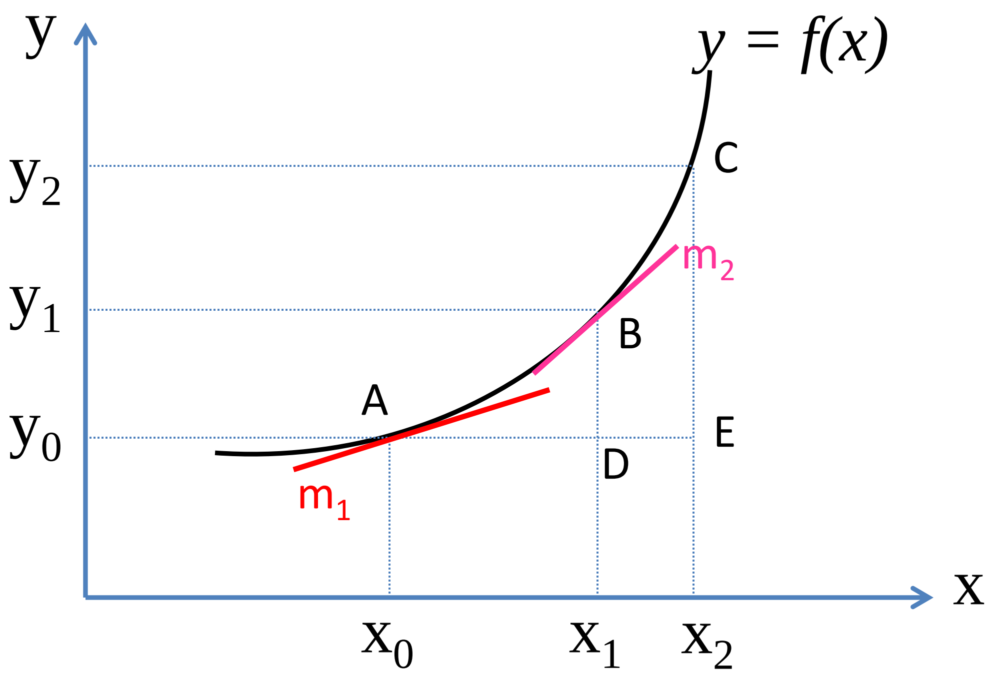
- Consider a linear function: $y = a + bx$.
- Graph



- Slope between point A and B is: $\dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{BC}{AC}$

Nonlinear Function

- Consider the following non-linear function



➢The slope between any two points vary along the curve.

➢The slope of the curve must be measured at the tangency of a particular point.

Rate of Change and the Derivative (1)

- Given the function *y = f(x)*, the difference quotient is the change in y per unit of change in x:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

➢ i.e. the difference quotient measures the average rate of change of y.

- Example: Given $y = f(x) = x^2 + 3x$, we can write:

$$f(x_0) = x_0^2 + 3x_0$$

$$f(x_0 + \Delta x) = (x_0 + \Delta x)^2 + 3(x_0 + \Delta x)$$

➔ $$\frac{\Delta y}{\Delta x} = 2x_0 + \Delta x + 3$$

Rate of Change and the Derivative (2)

- When *Δx is infinitesimally small,* the rate of change of y is called the derivative:

$$\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

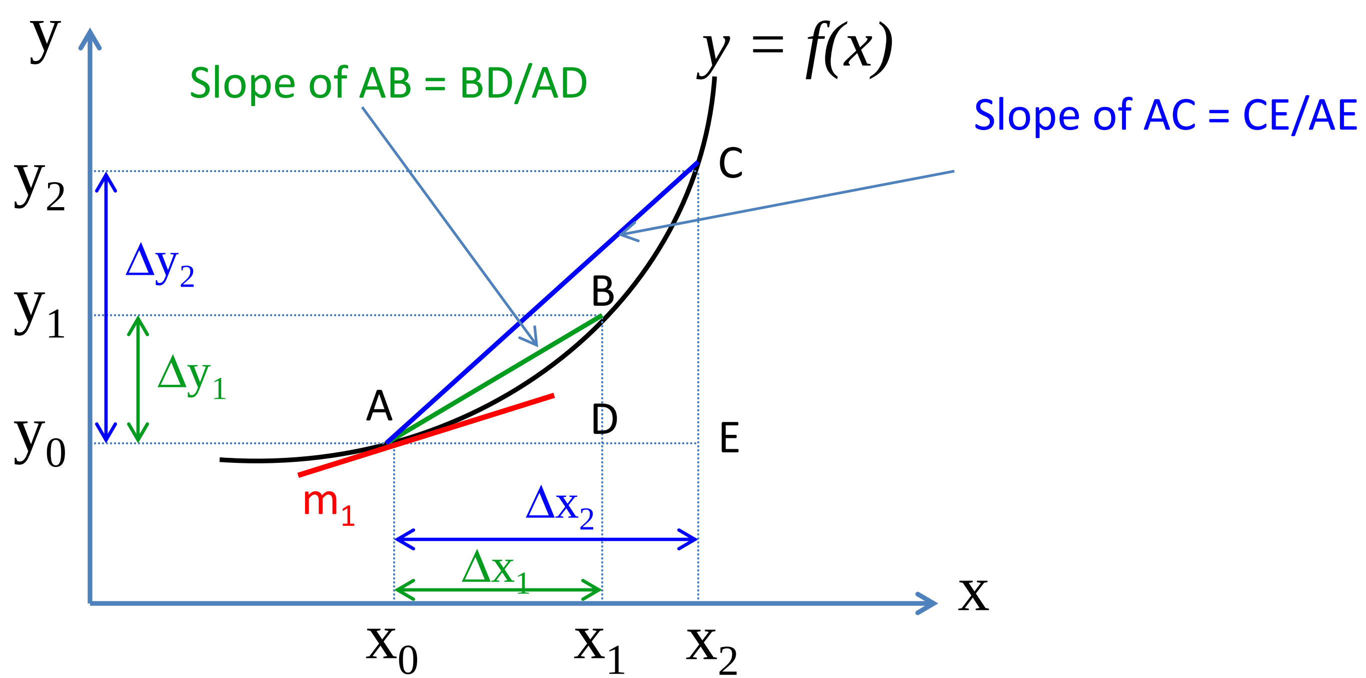
- i.e. the derivative of the function *y = f(x)* is the limit of the difference quotient *Δy/Δx*.

➢ The derivative measures the *instantaneous rate of change*.

- Example: Given $y = f(x) = x^2 + 3x$,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 2x_0 + \Delta x + 3 = 2x_0 + 3$$

Slope and Derivatives



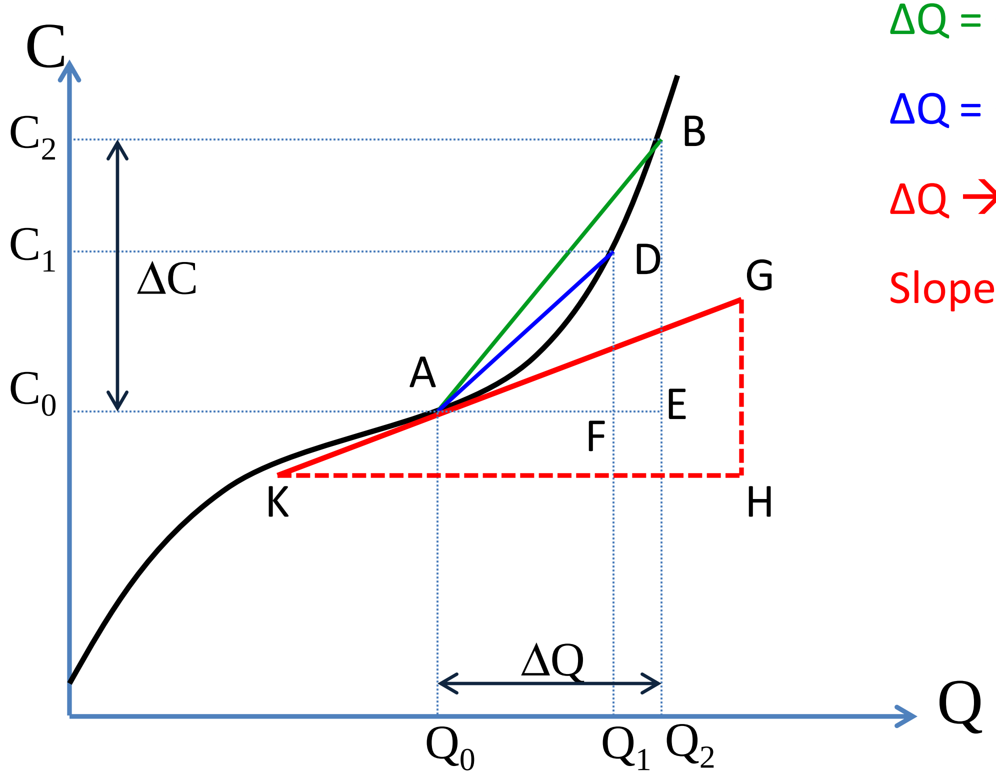
Slope at point A = Slope of m1

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The tangency to the curve *y = f(x)* at *x=a* is the slope: *m = f'(a).*

Slope and Derivatives

- Example: Total cost curve and its slope



$\Delta Q = Q_2 - Q_0$: Slope of AB= BE/AE

$\Delta Q = Q_1 - Q_0$: Slope of AD= DF/AF

$\Delta Q \to 0$:

Slope at A = Slope of KG (tangent at A)
= GH/KH
= $\lim_{\Delta Q \to 0} (\Delta C/\Delta Q)$
= $f'(Q_0)$

Differentiability of a Function

- The process of determining the derivative of a function is called "differentiation"
- A function is differentiable if it is continuous and smooth.
  - Continuity is a *necessary condition* for differentiability.
  - Smoothness is a *sufficient condition* for differentiability.
- Definition:

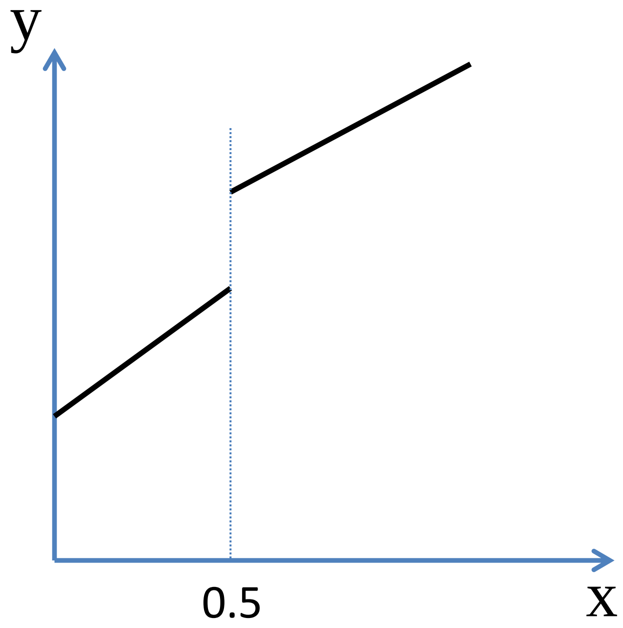
A function y = f(x) is continuous at x_0 if the followings are true:

1. $f(x_0)$ is defined.
2. $\lim_{x \to x_0}$ exists.
3. $\lim_{x \to x_0} f(x) = f(x_0)$

Continuity

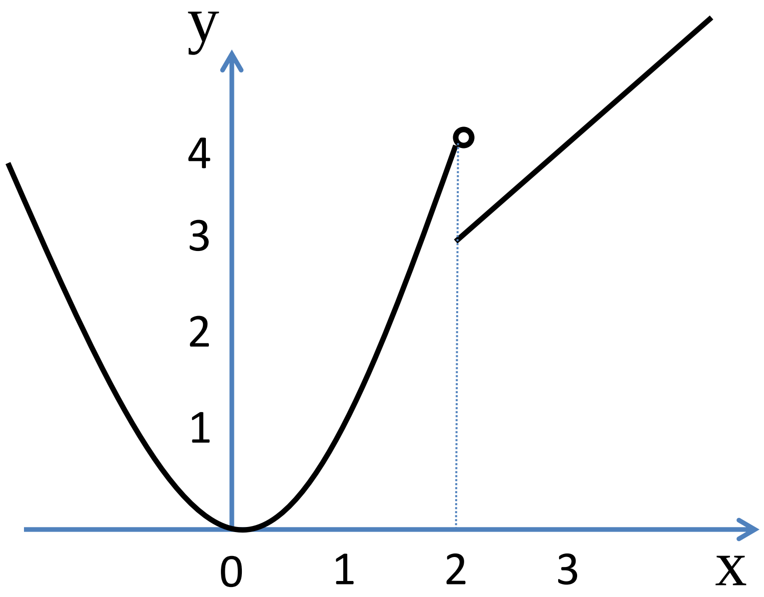
- Example 1:

f(x) is discontinuous at 0.5.



- Example 2:

$$g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x+1 & \text{if } x \geq 2 \end{cases}$$



Smoothness

- A function is smooth means that it is differentiable everywhere.
- Definition:

A function f(x) is differentiable at point *a* if

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Alternatively, given that *x = a + h*, then *x* → *a* if *h* → *0*. Then,

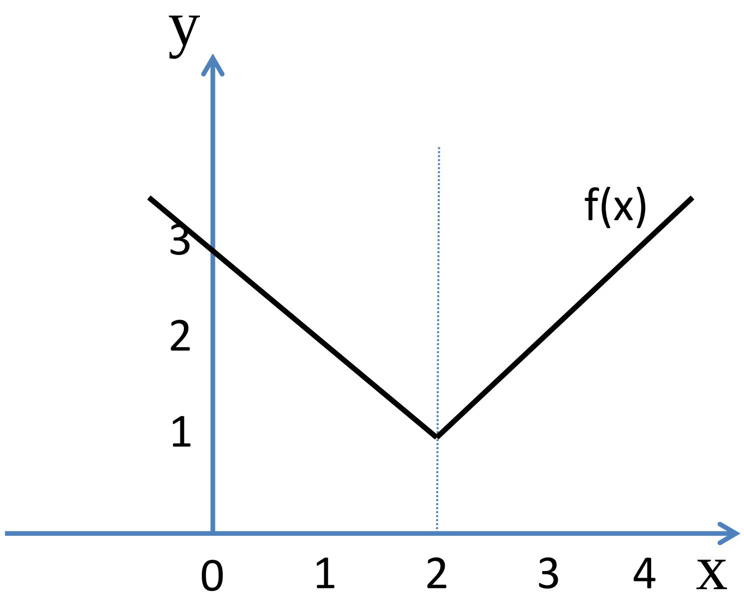
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, x \neq a$$

- Theorem:

If a function *f(x)* has a derivative at *a*, then it is continuous at *a* as well.

Non-Differentiable Functions

- Example: f(x) = |x – 2| + 1



$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = -1$$

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = 1$$

$$\therefore \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$ does not exist.

Thus, f(x) is not differentiable at x = 2.

Summary: All differentiable functions are continuous, but not all continuous functions are differentiable.

Rules of Differentiation:

A Function of One Variable

- Constant-Function Rule:

  $y = f(x) = c$, where c is a constant

  ➔ $f'(x) = 0$.

- Power-Function Rule:

  $y = f(x) = x^n$, where n is a real number

  ➔ $f'(x) = nx^{n-1}$

- Exponential-Function Rule:

  $y = f(x) = e^x$

  ➔ $f'(x) = e^x$

- Logarithm-Function Rule:

  $y = f(x) = \ln(x)$

  ➔ $f'(x) = 1/x$

Rules of Differentiation:

Two or More Functions of the Same Variable

- Sum-Difference Rule:

  $y = f(x) + g(x)$ ➔ $$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

  $y = f(x) - g(x)$ ➔ $$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

- Product Rule:

  $y = f(x)g(x)$ ➔ $$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x)$$

- Quotient rule:

  $$y = \frac{f(x)}{g(x)}$$

  ➔ $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

Examples

- Example 1:

  $f(x) = 2x^3 + 3x^2 - 5x + 1$

  $f'(x) = 6x^2 + 6x - 5$

- Example 2:

  $f(x) = 3x^{-2/3}$

  $f'(x) = -2x^{-5/3}$

- Example 3:

  $f(t) = e^{rt}$

  $f'(t) = re^{rt}$

Examples

- Example 4:

  $f(x) = (6x + 1)(x^2 - 2x)$

  $f'(x) = 18x^2 - 22x - 2$

- Example 5:

  $f(x) = (x + 1)/2x^2$

  $f'(x) = -(x + 2)/2x^3$, for $x \neq 0$

- Example 6:

  $f(x) = x^5(3x)^2$

  $f'(x) = 63x^6$

Rules of Differentiation:

Functions of Different Variables

- Chain Rule:

  z = f(y) and y = g(x) ➔ *z = f(g(x))*

  ➔ $$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)g'(x)$$

- Inverse-Function Rule:

  If the function *y = f(x)* represents a one-to-one mapping, the function *f* will have an inverse function: $x = f^{-1}(y)$.

  ➔ $$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Examples

- Example 7:

  $f(x) = (x^2 – 4x)^3$

  $f'(x) = 6x^5 – 60x^4 + 192x^3 – 192x^2$

- Example 8:

  $f(x) = \ln(2x^2)$

  $f'(x) = 2/x$

- Example 9:

  $f(x) = 3x^5 + x^2$

  $dx/dy = 1/(15x^4 + 2x)$

APPLICATIONS IN ECONOMICS

Derivative and Marginality

- In economics, *marginality* indicates a rate of change – *how much the value of the function changes as the choice variable (independent variable) increases by one unit*.
- Examples:

	Level	Average	Marginal
Production	TP=Q = g(L)	AP = TP/L	MP = d(TP)/dL
Revenue	TR = f(Q)	AR = TR/Q	MR = d(TR)/dQ MRP =d(TR)/dL
Cost	TC = h(Q)	AC = TC/Q	MC =d(TC)/dQ MFC =d(TC)/dL
Consumption	C	APC	MPC
Saving	S	APS	MPS
Investment	I		MPI

Total, Average, and Marginal Revenue

- Total Revenue: $TR = P \times Q$.
- Demand function: $Q_d = a - bP$.
  - Example: $Q_d = 250 - 10P$. ➔ $P = 25 - 0.1Q$
- TR = f(Q) = P(Q) x Q = (25 – 0.1Q) x Q

  ➔ $TR(Q) = 25Q - 0.1Q^2$

- AR = TR/Q = 25Q – 0.1Q ➔ AR(Q) = P(Q)
- MR = d(TR)/dQ = 25Q – 0.2Q
- From TR = P(Q) x Q and MR = d(TR)/dQ,

we have $MR = \frac{d}{dQ}[P(Q)xQ] = P(Q)\cdot\frac{dQ}{dQ} + Q.\frac{dP}{dQ}$

Thus, $\boxed{MR = AR + Q.\frac{dP}{dQ}}$ $, \because P(Q) = AR$

Elasticity, Total Revenue, and Marginal Revenue

- From above, $$MR = AR + Q.\frac{dP}{dQ}$$

$$MR = AR + Q.\frac{dP}{dQ} \cdot \frac{P}{P}$$

$$MR = AR + \frac{P}{\frac{dQ/Q}{dP/P}} = AR + \frac{P}{\varepsilon_d} = AR + \frac{AR}{\varepsilon_d}$$

$$\therefore MR = AR\left(1 + \frac{1}{\varepsilon_d}\right)$$

- If |Ed| > 1 (elastic), MR > 0
- If |Ed| < 1 (inelastic), MR < 0
- If |Ed| = 1 (unitary elastic), MR = 0

Example: Elasticity, MR, and AR

- Suppose $TR = 25Q - 0.1Q^2$

Total, Average, and Marginal Product

- Consider a short-run production: Q = g(L)
  - AP = Q/L = g(L)/L
  - MP = dQ/dL = d[g(L)]/dL = g'(L)
- Relation between AP and MP:

$$\frac{d(AP)}{dL} = \frac{d}{dL}\left[\frac{g(L)}{L}\right] = \frac{L \cdot g'(L) - g(L)(1)}{L^2} = \frac{1}{L}\left[g'(L) - \frac{g(L)}{L}\right] = \frac{1}{L}[MP(L) - AP(L)]$$

- Marginal Revenue Product: MRP

$$MRP = \frac{d}{dL}[P[Q(L)]xQ(L)]$$

[See additional notes for the derivation.]

From above, we can show that $MRP = MP(L)xMR(Q)$.

Total, Average, and Marginal Cost

- TC = FC + VC = a + f(Q) = C(Q)
  - AC = C(Q)/Q
  - MC = d(TC)/dQ = C'(Q)
- Relation between AC and MC:

$$\frac{d(AC)}{dQ} = \frac{d}{dQ}\left[\frac{C(Q)}{Q}\right] = \frac{Q \cdot C'(Q) - C(Q)(1)}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right] = \frac{1}{Q}[MC(Q) - AC(Q)]$$

- Marginal Factor Cost: MFC

$$MFC = \frac{d}{dL}[C(Q(L))] = \frac{dC}{dQ} \cdot \frac{dQ}{dL} = MC(Q)xMP(L)$$