Chapter 6: Function of several variables and Multivariate calculus

6.1) Function of several variables

Suppose we have two variables (x, y). A function that maps these two variables into another set valued of variable, z, is called function of two variables.

Example 6.A $f(x,y) = 2x + x^2y^3$ $z = x^{\frac{1}{2}} * y^{\frac{1}{2}}$

Similarly, a function that maps from the value of N-variable into another set valued of variable, z, is called function of N variables.

Example 6.B $f(x_1, x_2, \dots x_n) = x_1x_2x_3 \dots x_n$

Most economic functions require the representation of multivariate relationship.

Utility function: $U = f(x_1, x_2, x_3, \dots, x_n)$

Production function: y = f(K,L,...)

Demand function: $Q_x^d = f(P_x, I, P_y, T)$

Supply function: $Q_x^s = f(P_x, w, P_y)$

6.2) Multivariate differential calculus

6.2.1) Partial derivative

Definition: Suppose $y = f(x_1, x_2, x_3, \dots, x_n)$, i.e. $f: R^n \to R$, the partial derivative of "y" with respect to "x_i" is denoted by:

$$\boxed{\frac{\partial y}{\partial x_i}} = \frac{\partial f}{\partial x_i} = f_i = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

- Operationally, deriving the partial derivative of a function is so simple!!
- We simply treat all other variables, except x_i, as constant.
- Then, apply all the rule of differentiations that we knew from the case of single variable calculus to the multivariate function.
- Let's proceed to some examples.

Example 6.C:

Suppose that $U = -5x^2 - 12xy - 6y^2$ find $\frac{\partial U}{\partial x}$ $(= U_x)$ and $\frac{\partial U}{\partial y}$ $(= U_y)$.

$\frac{\partial u}{\partial x}$ (y as a constant)

first-order Partial derivative of the U function

$$\frac{\partial u}{\partial x} = -10 \cdot x - 12 \cdot y$$

$$\frac{\partial u}{\partial y} = -12 \cdot x - 12 \cdot y.$$

Graphical illustration of partial derivative

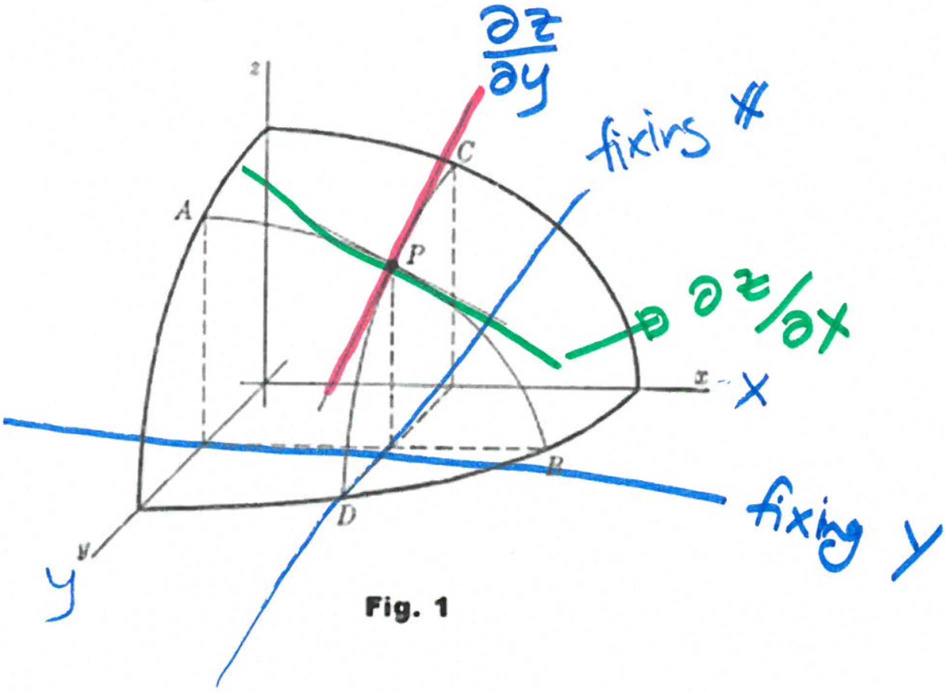

Fig. 1

Definition: *Gradient vector* (function)

Vector of the Collection of all the first-order partial derivative terms in matrix (vector)

$$\nabla U = \begin{bmatrix} U_x \\ U_y \end{bmatrix} = \nabla U = \begin{bmatrix} -10x - 12y \\ -12x - 12y \end{bmatrix}$$

For any arbitrary "n" variables, the gradient vector of $y = f(x_1, x_2, x_3, \dots, x_n)$ is given by

$$\nabla f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}_{n \times 1}$$

call Gradient vector.

Sol^n to
↳ optimization problem
x^* ~~inwhere~~ $\nabla f = 0$
inwhich

Example 6.C (cont): What is the second-order partial derivatives of the function used in Example 6.C above

$u_x = -10x - 12y \longrightarrow$

$u_y = -12x - 12y.$

$\underline{u_{xx}} = -10$

$u_{xy} = -12$

$u_{yx} = -12$

$\underline{u_{yy}} = -12$

u_x: slope of u along x axis

- $> 0 \rightarrow$ increasing along x axis
- $< 0 \rightarrow$ decreasing

$u_{xx} \Rightarrow$ how slope of u change over different value of x

Definition:

Hessian matrix: Collection of the second-order partial derivative in matrix form

$$H = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix}$$

if $U_{xx} > 0 \rightarrow$ Convex along x axis

if $U_{xx} < 0 \rightarrow$ Concave along x axis

Example 6.C (cont): Hessian matrix of U

$$H = \begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix} = \begin{pmatrix} -10 & \underline{-12} \\ \underline{-12} & -12 \end{pmatrix}$$

(i) Hessian is an N × N matrix. (square matrix)

(ii) a Symmetric matrix. $U_{xy} = U_{yx}$

<u>Young's theorem</u>

Short notation to represent the summation of² $\leftarrow \sum_{i=1}^{N} x_i = x_1 + x_2 + .. x_N$

Example 6.D: $f(x_1, x_2, \dots x_n) = x_1x_2x_3 \dots x_n = \prod_{i=1}^{n} x_i$ → short notation to represent product function

$f_1 = x_2x_3x_4 \dots x_n$

$f_2 = x_1x_3x_4 \dots x_n$

$\vdots$

$f_n = x_1x_2 \dots, x_{n-1}$

$$\nabla f = \begin{array}{cc} \begin{bmatrix} x_2x_3 .. x_n \\ x_1x_3 \dots x_n \\ \\ x_1x_2 .. x_{n-1} \end{bmatrix} & \begin{array}{c} \frac{f}{x_1} \\ \frac{f}{x_2} \\ \\ \frac{f}{x_n} \end{array} \end{array}$$

$$H = \begin{bmatrix} \quad \end{bmatrix}$$

Higher-order partial derivative

- Second-order partial derivative: $\left(\frac{\partial^2 f}{\partial x_i^2}\right)$

Differentiating the first-order derivative function with the same argument: $\frac{\partial\left(\frac{\partial f}{\partial x_i}\right)}{\partial x_i}$, i.e. differentiating the multivariate function with respect to the ***same*** argument twice.

- Second-order cross partial derivative: $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \frac{\partial\left(\frac{\partial f}{\partial x_i}\right)}{\partial x_j} = f_{ij}$

What does the second-order partial derivative tell us?

- Concavity/Convexity along a certain axis!

Notice an important property of the Hessian matrix

Hessian Matrix

For any arbitrary "n" variables, the ~~gradient vector~~ of y $= f(x_1, x_2, x_3, \dots, x_n)$ is given by

$$Df = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}$$

$$H = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix}_{n\times n}$$

$$\begin{matrix} f_{11} & f_{12} & f_{13} \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{matrix}$$

Example 6.D (cont): Hessian matrix of "f"

$f_{12} = f_{21}$

$$H = \begin{bmatrix} 0 & x_3x_4\dots x_n & x_2x_4\cdots x_n & \dots & x_2x_3\dots x_{n-1} \\ x_3x_4\dots x_n & 0 & \dots & & \\ & & & & \\ x_2x_3\dots x_{n-1} & \dots & & & 0 \end{bmatrix}$$

equal

$f_{1n} = f_{n1}$

Hessian Matrix of f

≡

Symmetric Matrix

What does partial derivative tell us in economics?

- Economically, what does the partial derivative, i.e. $\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_i\ (x_1, x_2, x_3, \dots, x_n)$, tell us about?
- Marginal effect of x_i on y. (marginal treatments)
  - Measuring the change in "y" that is purely attributed to the change in particular "x", while keeping all other "x" stay the same.
- This basically captures the concept so called "*Ceteris Paribus*".

, CRTS

$$t^1 \cdot K^{1/3} L^{2/3} = (tK)^{1/3} (tL)^{2/3} \leftarrow (tK, tL)$$

Example 6.E Consider a production function given by, $Q = K^{\frac{1}{3}} L^{\frac{2}{3}}$

a) Derive the expression for marginal product of labor (MPL) and capital (MPK), respectively

$$211 \rightarrow MPL = \frac{\Delta Q}{\Delta L}$$

$$\frac{\partial^2 Q}{\partial L^2} = \frac{2}{3}\left(-\frac{1}{3}\right) \underbrace{K^{\frac{1}{3}} L^{-\frac{4}{3}}}_{>0} < 0$$

$$320 \rightarrow MPL = \frac{\partial Q}{\partial L} = \frac{2}{3} K^{1/3} L^{-1/3}$$

$$\frac{\partial \frac{\partial Q}{\partial L}}{\partial K} = \frac{2}{3} \cdot \frac{1}{3} K^{-2/3} L^{-1/3} \rightarrow > 0$$

mpL function (MPL function ; K, L)

Production f^2 ⇒ Law of diminishing Marginal productivity

$$MPK = \frac{\Delta Q}{\Delta K} \rightarrow MPK = \frac{\partial Q}{\partial K}$$

$$= \frac{1}{3} K^{-2/3} \cdot L^{2/3}$$

MPK Function in terms of K and L.

→ Reason why US wage > Thai Wage is attributed to difference in the ~~land~~ level of K.

b) How does the MPK change with respect to k? Similarly, how does the MPL change with respect to L?

c) How does the MPL change with respect to k? Discuss the implication of you result.

Cobb-Douglass function

- Charles W. Cobb and Paul H. Douglas (1928) *"A Theory of Production"* AER: cited as one of the top 20 most influential paper in economics
- General form: $Q = K^{\alpha}L^{\beta}$;
  - In our example 6.E, $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$
- Mathematically, the Cobb-Douglass production function is a **homogenous function**.
  - $\alpha + \beta$ is the degree of homogeneity.

Definition: A function is said to be a homogenous function if

$$f(tx_1, tx_2, \dots, tx_n) = t^m f(x_1, x_2, \dots, x_n)$$

where "m" is called the degree of homogeneity.

The Cobb-Douglass function is a homogenous function

- **Proof:** $(tK)^{\alpha}(tL)^{\beta} = t^{\alpha+\beta}K^{\alpha}L^{\beta}=t^{\alpha+\beta}Q$

Example 6.F: Which one is HM function, and to what degree.

- $z = x^2 + y^2$ # 2
- $z = x + y^2$ not homogeneous f^n
- $z = x^2 y + x^{\frac{8}{3}} y^{\frac{1}{3}}$ # 3
- $z = (\alpha K^{1-a} + \beta L^{1-a})^{\frac{1}{1-a}}$ # 1

↳ CES Production f^n $\Downarrow$

Mathematical properties of HM function of degree "m".

(i) f_i is degree of homogeneity "m-1".

(ii) Euler theorem: $\sum_{i=1}^{N} f_i x_i = m f(x_1, x_2, \ldots, x_N)$

Why do we use HM function in economics?

- For the production, HM reflects the assumption that production technology exhibits return to scale.
- What is the return to scale?
  - If you *proportionately* increase "K" and "L", how much is the increase in "Q", measuring in terms of its proportion to an increase in factor input.

$$(x_0, y_0) \rightarrow z_0 = (x_0^2 y_0) + (x_0^{8/3} y_0^{1/3})$$

$$\underline{\underline{t}}(x_0, y_0) \rightarrow z^{new} = (t_0 x_0)^2 (t_0 y_0) + (t_0 x_0)^{8/3} (t_0 y_0)^{1/3}$$

$$\Downarrow$$

$$\overset{\curvearrowleft\curvearrowright}{t_0^2} x_0^2 \cdot t_0 y_0 + t_0^{8/3} x_0^{8/3} \cdot t_0^{1/3} y_0^{1/3}$$

$$t_0^3 x_0^2 \cdot y_0 + t_0^{8/3 + 1/3} \cdot x_0^{8/3} y_0^{1/3}$$

$$= t_0^3 \Big(\underbrace{x_0^2 \cdot y_0 + x_0^{8/3} y_0^{1/3}}_{z_0} \Big)$$

$$z^{new} = t_0^{\textcircled{3}} \cdot z_0 \rightarrow$$ HM f^2 with degree #3.

$$z = x^2 + y^2 \longrightarrow z_0 = x_0^2 + y_0^2$$

$$\left.\begin{array}{l} \underline{x_0} \rightarrow \\ \underline{y_0} \rightarrow \end{array} \boxed{\begin{array}{c} t x_0 \\ t y_0 \end{array}}\right\} \Rightarrow z^{new}$$ compared to z_0

$$\rightarrow \quad z^{new} = (t x_0)^2 + (t y_0)^2$$

$$\rightarrow \quad = t^2 x_0^2 + t^2 \cdot y_0^2 = t^2 \underbrace{(x_0^2 + y_0^2)}_{z_0}$$

$$z^{new} = t^2 \cdot z_0$$

(i) hm f^2 (ii) degree of the homogeniety is 2.

$m = 1$; f: productn f^n

$f(K, L)$

$$\Rightarrow \quad \frac{\partial f}{\partial K} \cdot K + \frac{\partial f}{\partial L} \cdot L = (1) \cdot f(K, L)$$

Euler's Theorem $\Rightarrow$ Income Distribution

$K^{\alpha} L^{\beta}$

$\textcircled{\alpha}$ → share of Capital income in the market Economy

β → share of labor income in the market Economy

$$Q = K^{\alpha} \cdot L^{\beta} \longrightarrow Q = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$$

→ Function is a Homogeneous f^2

A function is called HM f^2 if.

degree of homogeniety of f^2

$$f(tx_1, tx_2, \ldots, tx_n) = t^{\enclose{circle}{r}} \cdot f(x_1, x_2, \ldots; x_n)$$

$$\begin{pmatrix} K \\ L \end{pmatrix} \rightarrow \begin{pmatrix} tk \\ tL \end{pmatrix} \Rightarrow \text{new } Q = (tK)^{\alpha} (tL)^{\beta}$$

$$= \enclose{circle}{t^{\alpha+\beta}} \cdot K^{\alpha} \cdot L^{\beta}$$

Our Cobb Douglass $f^2 \Longrightarrow$ HM f^2 Degree $\alpha + \beta$

Type of Production f^n

(1) Short-run : fixed input variable

(2) Long-run : all the input can be changed

$Q = f(K, L)$

If both K and L can be changed → LR production.

one of them is fixed ⇒ SR production

$Q = K^{\frac{1}{3}} L^{2/3}$ ⇒ Cobb Douglass function

Return to Scale

③ In the Cobb Douglas production f^n

$\alpha + \beta = $ HM of f^n

our case $\alpha + \beta = \frac{1}{3} + \frac{2}{3} = 1$

IRIS

① Increasing Return to Scale. $r > 1$

if ⇒ double inputs ⇒ output increases more than double

CRTS

② Constant Return to Scale $r = 1$

Double inputs ⇒ Double output.

DRTS

③ Decreasing Return to Scale $r < 1$

Double inputs but less then double of the output earned.

Return to Scale is linked to the degree of the homogeneity of the production function (r)

- Degree of homogeneity can tell us:

$$\alpha + \beta = 1: \text{ Constant return to scale}$$

$$\alpha + \beta > 1: \text{ Increasing return to scale}$$

$$\alpha + \beta < 1: \text{ Decreasing return to scale}$$

Elasticity versus Partial elasticity

- $y = f(x)$ ➔ Elasticity $= \frac{dy}{dx} * \cancel{\frac{y}{x}} = \frac{d\ln(y)}{d\ln(x)}$

$\frac{dy}{dx} \cdot \frac{x}{y}$

- $z = f(w, x)$
  - Partial elasticity of z with respect on w

$$= \frac{\partial z}{\partial w} * \frac{w}{z} = \frac{\partial \ln(z)}{\partial \ln(w)}$$

  - Partial elasticity of z with respect on x

$$= \frac{\partial z}{\partial x} * \frac{x}{z} = \frac{\partial \ln(z)}{\partial \ln(x)}$$