Due date: February 8, 2022 before 2.00 pm

Question 1 (60 Points) Score....................

Consider the individual's portfolio choice problem given in the below equation:

$$\max_A E[U(\tilde{W})] = \max_A E[U(W_0(1+r_f) + A(\tilde{r} - r_f))]$$

Assume the utility of this investor: $U(W) = ln(W)$ and the rate of return on the risky asset equals

$$\tilde{r} = \begin{cases} 4r_f & \text{with probability} \frac{1}{2} \\ -r_f & \text{with probability} \frac{1}{2} \end{cases}$$

Solve for the individual's proportion of initial wealth invested in the risky asset, $(\frac{A}{W_0})$.

Optimal level of A

As $u(w) = ln(w)$, we can get first order conditions as

F.O.C. : $\frac{\partial E[u(\tilde{w})]}{\partial A} = E[u'(\tilde{w})(\tilde{r} - r_f)] = 0$

$\tilde{r} = 4r_f \Leftarrow$ high

$= \frac{\partial u(\tilde{w})}{\partial \tilde{w}} \cdot (4r_f - r_f)$

$= \left[\frac{1}{w_0(1+r_f) + A(3r_f)} \cdot 3r_f\right]$

$= \frac{3r_f}{w_0(1+r_f) + 3r_f A}$

$\tilde{r} = -r_f$ ⇐ low

$$= \left[\frac{1}{w_0(1+r_f)+A(-2r_f)} \cdot (-2r_f)\right]$$

$$= \frac{-2r_f}{W_0(1+r_f)-2r_fA}$$

According to FOC ⇒ $\frac{1}{2}\left[\frac{-2r_f}{w(1+r_f)-2r_fA}\right] + \frac{1}{2}\left[\frac{3r_f}{w_0(1+r_f)+3r_fA}\right] = 0$

$$\Rightarrow \quad \frac{3}{2}\,\frac{\cancel{r_f}}{W_0(1+r_f)+3r_fA} = \frac{\cancel{r_f}}{W_0(1+r_f)-2r_fA}$$

$$3W_0 + 3W_0r_f - 6r_fA = 2W_0 + 2W_0r_f + 6r_fA$$

$$W_0 + W_0r_{f_0} = 12r_fA$$

$$W_0(1+r_{f_0}) = 12r_fA$$

$$\therefore \frac{A}{W_0} = \frac{(1+r_{f_0})}{12r_f}$$

Question 2 (60 Points) Score....................

An expected-utility-maximizing individual has constant relative-risk-aversion utility,

$$U(W) = \frac{W^{\gamma}}{\gamma}$$

$U(W) = -W^{-1}$
$U'(W) = W^{-2}$

,with relative-risk-aversion coefficient of $\gamma = -1$. The individual currently owns a product that has a probability p to failing,an event that would result in a loss of wealth that has a present value equal to L. With probability 1-p, the product will not fail and no loss will result. The individual is considering wherther to purchase an extended warranty on this product. The warranty costs C and would insure the individual against loss if the product fails. Assuming that the cost of the warranty exceeds the expected loss from the product's failure, determine the individual's level of wealth at which she would be just indifferent between purchasing or not purchasing the warranty.

Loss = L with prob. = p   not fail prob. = 1-p

Cost of Waranty = C

$$U(W-C) = E[U(W+\tilde{L})]$$

$\gamma = -1$ $U(W) = -W^{-1}$

$$\frac{-1}{(W-C)} = p\left(-\frac{1}{W-L}\right) + (1-p)\left(-\frac{1}{W}\right)$$

$$\frac{-1}{W-C} = \frac{-p}{W-L} - \frac{1}{W} + \frac{p}{W}$$

$$\left[\frac{p}{W-L} - \frac{1}{W-C}\right]W = -1 + p$$

$$\frac{Wp(W-C) - W(W-L)}{(W-L)(W-C)} = -1+p$$

$$\cancel{W^2p} - \cancel{WpC} - \cancel{W^2} + \cancel{WL} = -\cancel{W^2} + \cancel{WL} + WC - LC + \cancel{pW^2} + WpL - \cancel{WpC} + pCL$$

$$0 = WC - LC + WpL + pCL$$

$$LC - pCL = W(C + pL)$$

$$\therefore W = \frac{LC - pCL}{C + pL}$$ ※

Question 3 (60 Points) Score.....................

Risk Aversion: Consider the following utility functions (Defined over wealth:W)

(1) U(W) =$-\frac{1}{W}$
(2) U(W) =ln(W)
(3) U(W) =-$W^{-\gamma}$
(4) U(W) =-exp(-γW)
(5) U(W) =$\frac{W^{\gamma}}{\gamma}$
(6) U(W) =αW-βW^2

Questions:

(a) Check that they are well behaved (U'>0 and U"<0) or state restriction on the parameters so that they are. For the utility function (6), take the positive α and β , and give the range of wealth over which the utility function is well behaved.

(b) Compute the absolute and relative risk aversion coefficients.

(c) What is the effect of parameter α (when relevant)?

(d) Classify the functions as increasing /decreasing risk aversion utility functions (both absolute and relative).

1) $U(W) = -\frac{1}{W}$ a) $U'(W) = \frac{1}{W^2} > 0$ $U''(W) = \frac{-2}{W^3} < 0$; $W > 0$

b) $R(W) = -U''(W)/U'(W) = \frac{2}{W^3} \cdot \frac{W^2}{1} = \frac{1}{W}$

$R_r(W) = W \cdot R(W) = 1$

d) Absolute $\frac{\partial R(W)}{\partial (W)} = \frac{-1}{W^2} < 0$ → decreasing in Absolute

Relative $\frac{\partial R_r(W)}{\partial W} = 0$ → constant

2) $U(W) = \ln(W)$ a) $U'(W) = \frac{1}{W} > 0$ $U''(W) = -\frac{1}{W^2} < 0$; $W > 0$

b) $R(W) = -U''(W)/U'(W) = \frac{1}{W^2} \cdot W = \frac{1}{W}$

$R_r(W) = W \cdot R(W) = W \cdot \frac{1}{W} = 1$

d) Absolute $\frac{\partial R(W)}{\partial W} = -\frac{1}{W^2} < 0$; decreasing in absolute

Relative $\cdot \frac{\partial R_r(W)}{\partial W} = 0 \rightarrow$ constant

3) $U(W) = -W^{-\gamma}$ a) $U'(W) = \gamma W^{-\gamma-1} > 0$ $U''(W) = \underbrace{(-\gamma-1)}_{<0}\underbrace{(\gamma)}_{>0} \underbrace{W^{-\gamma-2}}_{>0} < 0$; $0 < \gamma \leq 1$

b) $R(W) = -U''(W)/U'(W) = -\frac{(-\gamma-1)(\cancel{\gamma})W^{-\gamma-2}}{\cancel{\gamma} W^{-\gamma-1}} = (\gamma+1)(W^{-1})$

$R_r(W) = W \cdot R(W) = \gamma + 1$

d) Absolute $\rightarrow \frac{\partial R(W)}{\partial W} = (-\gamma-1)(W^{-2})$; decreasing when $\gamma < 1$, increasing when $\gamma > -1$

Relative $\rightarrow \frac{\partial R_r(W)}{\partial(W)} = 0 \rightarrow$ constant

4) $U(W) = -\exp(-\gamma W) = -e^{(-\gamma W)}$ a) $U'(W) = \gamma e^{(-\gamma W)} > 0$ b) $U''(W) = -[\gamma^2 e^{(-\gamma W)}] < 0$; $0 < \gamma \leq 1$

b) $R(W) = -U''(W)/U'(W) = \gamma^2 e^{(-\gamma W)} / -e^{(-\gamma W)} = -\gamma^2$

$R_r(W) = W \cdot R(W) = -W\gamma^2$

d) Absolute $\rightarrow \frac{\partial R(W)}{\partial(W)} = 0 \rightarrow$ constant Relative $\cdot \frac{\partial R_r(W)}{\partial(W)} = -\gamma^2 < 0$; decreasing in relative

5) $U(W) = \frac{W^\gamma}{\gamma}$ a) $U'(W) = \frac{\cancel{\gamma} W^{\gamma-1}}{\cancel{\gamma}} > 0$ b) $U''(W) = \underbrace{(\gamma-1)}_{<0} \underbrace{W^{\gamma-2}}_{>0} < 0$; $0 < \gamma \leq 1$

b) $R(W) = -U''(W)/U'(W) = (-\gamma+1)W^{\gamma-2} / W^{\gamma-1} = (-\gamma+1)W^{-3}$

$R_r(W) = W \cdot R(W) = (-\gamma+1)W^{-2}$

d) Absolute $\rightarrow \frac{\partial R(W)}{\partial(W)} = (3\gamma-3)W^{-4}$; $0 < \gamma < 1$; decreasing in absolute

Relative $\rightarrow \frac{\partial R_r(W)}{\partial(W)} = (2\gamma-2)W^{-3}$; $0 < \gamma < 1$; decreasing in relative.

6) $U(W) = \alpha W - \beta W^2$ a) $U'(W) = \alpha - 2\beta W > 0$ b) $U''(W) = -2\beta < 0$; $\beta > 0, \alpha > 2\beta W$

b) $R(W) = -U''(W)/U'(W) = 2\beta/\alpha - 2\beta W$ $R_r(W) = W \cdot R(W) = 2\beta W / \alpha - 2\beta W$

c) Effect of α : It can change the function to be increasing or decreasing function depends on α's size.

Effect of β: Similar to α but also change the function to concave or convex.

d) Absolute $\rightarrow \partial R(W)/\partial W = [(\alpha - 2\beta W) - 2\beta(-2\beta)]/(\alpha-2\beta W)^2 = \frac{\alpha - 2\beta W + 4\beta^2}{(\alpha-2\beta W)^2} > 0 \rightarrow$ increasing in absolute

Relative $\rightarrow \partial R_r(W)/\partial W = [(\alpha - 2\beta W)(2\beta) - 2\beta W(-2\beta)]/(\alpha-2\beta W)^2 = \frac{2\alpha\beta + 4\beta^2 W + 4\beta^2 W}{(\alpha-2\beta W)^2} > 0 \rightarrow$ increasing in relative.