R^2 in Model with an intercept

$$Y_i = \alpha + \beta X_i + u_i$$

$$\therefore \; TSS = ESS + RSS$$

$$\Sigma y_i^2 = \Sigma \hat{y}_i^2 + \Sigma \hat{u}_i^2$$

Divide through by Σy_i^2, we obtain

$$1 = \frac{\Sigma \hat{y}_i^2}{\Sigma y_i^2} + \frac{\Sigma \hat{u}_i^2}{\Sigma y_i^2}$$

$$1 = R^2 + \frac{\Sigma \hat{u}_i^2}{\Sigma y_i^2}$$

$$\text{or } R^2 = 1 - \frac{\Sigma \hat{u}_i^2}{\Sigma y_i^2}$$

The minimum value of $R^2 = 0$, which will occur only when $\Sigma \hat{u}_i^2 = \Sigma y_i^2$ or when X cannot explain the variation in Y at all which means $\beta = 0$.

Therefore R^2 in model with an intercept will lie between 0 and 1.

R^2 in Model through the origin (without an intercept)

$$Y_i = \beta X_i + u_i$$

OLS : $\hat{\beta} = \dfrac{\sum X_i Y_i}{\sum X_i^2}$

$$Y_i = \hat{Y}_i + \hat{u}_i \quad \text{where} \quad \hat{Y}_i = \hat{\beta} X_i$$

$$Y_i = \hat{\beta} X_i + \hat{u}_i$$

$$Y_i^2 = \hat{\beta}^2 X_i^2 + \hat{u}_i^2 + 2\hat{\beta} X_i u_i$$

$$\sum Y_i^2 = \hat{\beta}^2 \sum X_i^2 + \sum \hat{u}_i^2 + 2\hat{\beta} \underbrace{\sum X_i u_i}_{=0}$$

Therefore

$$\sum Y_i^2 - \hat{\beta}^2 \sum X_i^2 = \sum \hat{u}_i^2$$

Sum of squares $\sum Y_i^2$ and $\sum X_i^2$ are not mean adjusted.

$$R^2 = 1 - \frac{\sum \hat{u}_i^2}{\sum y_i^2}$$

$$= 1 - \frac{\left(\sum Y_i^2 - \hat{\beta}^2 \sum X_i^2\right)}{\sum y_i^2}$$

It can be shown that $\sum y_i^2 = \sum Y_i^2 - n\bar{Y}^2$.

$$R^2 = 1 - \frac{\left(\sum Y_i^2 - \hat{\beta}^2 \sum X_i^2\right)}{\sum Y_i^2 - n\bar{Y}^2}$$

There is no guarantee that $\hat{\beta}^2 \sum X_i^2 > n\bar{Y}^2$. Therefore it is possible that R^2 can be negative.