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8. Multiple Regression Analysis: The Problem of Inference

In this chapter, we will extend the ideas of interval estimation and hypothesis testing developed there to models involving three or more variables.

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

We have already known that if our objective is to do interval estimation and hypothesis testing, we need to assume that the u_i follow the normal distribution with zero mean and constant variance σ^2

With the normality assumption and the CLRM assumptions, we know that:

[1] The OLS estimations of partial regression coefficients are best linear unbiased estimators (BLUE).

[2] The estimators $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\beta}_3$ are normally distributed with means equal to true β_1,β_2, and β_3 and variances are following:

$$var(\hat{\beta}_1) = \left[\frac{1}{n} + \frac{\bar{X}_2^2 \sum x_{3i}^2 + \bar{X}_3{}^2 \sum x_{2i}^2 - 2\bar{X}_2\bar{X}_3 \sum x_{2i}x_{3i}}{\sum x_{2i}^2 \sum x_{3i}^2 - (\sum x_{2i}x_{3i})^2}\right] * \sigma^2$$

$$se(\hat{\beta}_1) = +\sqrt{var(\hat{\beta}_1)}$$

$$var(\hat{\beta}_2) = \frac{\sum x_{3i}^2}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i}x_{3i})^2} * \sigma^2$$

$$var(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2(1 - r_{23}^2)}$$

$$se(\hat{\beta}_2) = +\sqrt{var(\hat{\beta}_2)}$$

$$var(\hat{\beta}_3) = \frac{\sum x_{2i}^2}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i}x_{3i})^2} * \sigma^2$$

$$var(\hat{\beta}_3) = \frac{\sigma^2}{\sum x_{3i}^2(1 - r_{23}^2)} \longrightarrow \hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{(n-3)}$$

$$se(\hat{\beta}_3) = +\sqrt{var(\hat{\beta}_3)}$$

Moreover, $\frac{(n-3)\hat{\sigma}^2}{\sigma^2}$ follows the χ^2 distribution with n-3 df. We can also show that, if we replace the true σ^2 by its unbiased estimator $\hat{\sigma}^2$ in the computation of the standard errors, we then get

$$t = \frac{\hat{\beta}_1 - \beta_1}{se(\hat{\beta}_1)}$$

$$t = \frac{\hat{\beta}_2 - \beta_2}{se(\hat{\beta}_2)}$$

$$t = \frac{\hat{\beta}_3 - \beta_3}{se(\hat{\beta}_3)}$$

follows the t distribution with n-3 df.

Example Consider the following regression:

$$\begin{aligned} \log(\widehat{\text{salary}}) &= 4.32 + 0.280 \ \log(\text{sales}) + 0.0174 \ \text{ROE} + 0.00024 \ \text{ROS} \\ se &= (0.32) \ (0.035) \qquad (0.0041) \qquad (0.00054) \end{aligned} \tag{8.1}$$

$$R^2 = 0.283$$

where
salary =salary of CEO (thousand of dollars)
sales=annual firm sales
ROE=return on equity in percent → Ex: ROE = 10, then the average return on equity is 10%
ROS =return on firm's stock

Interpreter the partial regression coefficients

Questions What about the statistical significance of the observed results?

Questions What about the statistical significance of the observed results?

For the coefficient of log(sales) of 0.280, Is this coefficient statistically significant different from zero?

For the coefficient of ROE of 0.0174, Is this coefficient statistically significant different from zero?

For the coefficient of ROS of 0.00024, Is this coefficient statistically significant different from zero?

Are these three coefficients statistically significant?
To answer these questions, we have to learn the kinds of hypothesis testing.

8.1 Hypothesis Testing About Individual Regression Coefficients

We can use the t-test to test a hypothesis about any individual partial regression coefficient.

8.1.1 Two-tail test:

Let us postulate that

$H_0: \ \beta_2 = 0$ (sales does not affect CEO salary)
$H_1: \ \beta_2 \neq 0$ (sales does affect CEO salary)

suppose $\alpha = 0.05$

① compute $\hat{t}$:

$$\hat{t} = \frac{\hat{\beta}_2 - \beta_2}{se_{\hat{\beta}_2}} = \frac{0.28 - 0}{0.035} = 8.00$$

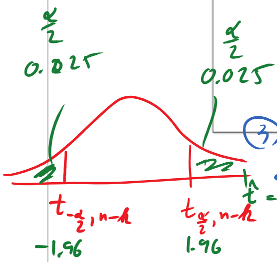
② Find critical t :

✓ $n = 209$

✓ $df = n - k = 209 - 4 = 205$

$\frac{\alpha}{2}$ 0.025

$$t_{\frac{\alpha}{2}, n-k} = t_{0.025, 205} = 1.96$$



③ compare $\hat{t}$ w/ critical t :

since $\hat{t} > t_{CRITICAL}$, then we may reject the null hypothesis that $H_0: \beta_2 = 0$.

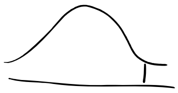
In otherwords, $\hat{\beta}_2$ is statistically significant different from zero at 95%

8.1.2 One-tail test:

Let us postulate that

$H_0: \ \beta_2 \leq 0$ (sales negatively impact CEO salary)

$H_1: \ \beta_2 > 0$ (sales positively impact CEO salary)



D-I-Y

$t_{\alpha, df} = t_{0.05, 205} = 1.645$

8.2 Testing The Overall Significance of the Sample Regression (JOINT TEST)

In the previous section, we test the significance of the estimated partial regression coefficients individually, that is under the separate hypothesis that each true population partial regression coefficient was zero. But now we are interested in testing β_2 , β_3 and β_4 are jointly or simultaneously equal to zero. In other words, we would like to test the following hypothesis:

$$\boxed{H_0: \beta_2 = \beta_3 = \beta_4 = 0}$$

H_1: At least one of them is not equal to zero

In order to reach this goal, we have to learn the following test.

The Analysis of Variance Approach to Testing the Overall Significance of an Observed Multiple Regression: The F-Test

The joint hypothesis can be tested by the **Analysis of Variance (ANOVA)** which can be demonstrated as follows:

($k=3$)

Table 8.1: ANOVA Table for the three-variable regression model

Source of variation	Sum of Square **SS**	**df**	Mean Sum of Square **MSS**
Due to regression (ESS)	$\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i}$	2	$\frac{\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i}}{2}$
Due to residuals (RSS)	$\sum \hat{u}_i^2$	$n-3$	$\frac{\sum \hat{u}_i^2}{n-3}$
TSS	$\sum y_i^2$	$n-1$	$\frac{\sum y_i^2}{n-1}$

$$TSS = ESS + RSS$$

$$\sum y_i^2 = \hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i} + \sum \hat{u}_i^2$$

FOR JOINT TEST, [$H_0: \beta_2 = \beta_3 = 0$], WE COMPUTE F-STATISTIC:

$$F = \frac{ESS/df}{RSS/df} = \frac{(\hat{\beta}_2 \sum x_{2i} y_i + \hat{\beta}_3 \sum x_{3i} y_i)/2}{\sum \hat{u}_i^2 / (n-3)}$$

F is distributed w/ 2 and $n-3$ df.

To compute $\sum \hat{u}_i^2$ we have to estimate $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ which consume 3 d.f. So, $df = n-3$.

Decision Rule Given the k- variable regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + ... + \beta_k X_{ki} + u_i$$

To test the hypothesis

$$H_0: \ \beta_2 = \beta_3 = ... = \beta_k = 0$$

(i.e ., all slope coefficients are simultaneously zero) versus

H_1 Not all slope coefficients are simultaneously zero

If $F > F_\alpha(k-1, n-k)$, we reject H_o ; otherwise we cannot reject it, where $F_\alpha(k-1, n-k)$ is the critical F value at the α level of significance and (k-1) numerator df and (n-k) denominator df.

An important Relationship between R^2 and F

$$F = \frac{ESS/(k-1)}{RSS/(n-k)}$$

\# of regressors + intercept

$$F = \frac{(n-k)}{(k-1)} \cdot \frac{ESS}{RSS}$$

$$= \frac{(n-k)}{(k-1)} \cdot \frac{ESS}{TSS - ESS}$$

$$= \frac{(n-k)}{(k-1)} \cdot \frac{\frac{ESS}{TSS}}{\frac{TSS-ESS}{TSS}}$$

$$= \frac{(n-k)}{(k-1)} \cdot \frac{R^2}{1-R^2}$$

$$*** \quad F = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \Rightarrow$$

① If $R^2 = 0$, $F = 0$

② The higher the R^2, The greater the F value

③ In limit, when $R^2 = 1$, F is infinite.

Ex: $H_0: \beta_2 = \beta_3 = \beta_4 = 0$ (CEO SALARY MODEL)

H_1: Not all slope coefficients are zero simultaneously

STEP 1 COMPUTE F-STATISTIC

$$F = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} = \frac{0.28/(4-1)}{(1-0.28)/(209-4)} = 26.9712$$

STEP 2 FIND CRITICAL F STAT:

$\alpha = 0.05$, $F_{0.05,3,205} \approx 2.65$.

STEP 3 COMPARE $F_{COMPUTED}$ W/ $F_{CRITICAL}$

since $\hat{F} = 26.9712 > F_{CRITICAL} = 2.65$,

then we may reject $H_0: \beta_2 = \beta_3 = \beta_4 = 0$ at 95% confidence level.

Table 8.2: ANOVA Table in Terms of R^2

Source of variation	Sum of Square SS	df	Mean Sum of Square MSS
Due to regression (ESS)	$R^2 \cdot \sum y_i^2$	2	$R^2 \cdot \sum y_i^2 / 2$
Due to residuals (RSS)	$(1-R^2) \cdot \sum y_i^2$	$n-3$	$(1-R^2) \cdot \sum y_i^2 / (n-3)$
TSS	$\sum y_i^2$	$n-1$	

Decision Rule Testing the overall significance of a regression in terms of R^2

Given the k- variable regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + ... + \beta_k X_{ki} + u_i$$

To test the hypothesis

$$H_0: \ \beta_2 = \beta_3 = ... = \beta_k = 0$$

(i.e ., all slope coefficients are simultaneously zero) versus

H_1 Not all slope coefficients are simultaneously zero

Compute

$$\boxed{F = \frac{R^2/(k-1)}{(1-R^2)/(n-k)}}$$

If $F > F_\alpha(k-1, n-k)$, we reject H_o ; otherwise we cannot reject it, where $F_\alpha(k-1, n-k)$ is the critical F value at the α level of significance and (k-1) numerator df and (n-k) denominator df.

8.3 The "Incremental" or "Marginal" Contribution of an Explanatory Variable

Let consider the following regression:

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + u_i$$ → model 1

Having run the above regression, let us suppose we decide to add the additional variable, X_{3i}, to the model and obtain the multiple regression as follow:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$ → model 2

Comparing between these two regressions, we might need to answer the below questions:

[1]. What are the marginal, or incremental, contribution of X_{3i}, knowing that X_{2i} is already in the model and that it is significantly related to Y_i.

[2]. Is the incremental contribution of X_{3i} statistically significant?

[3]. What is the criterion for adding variables to the model?

By contribution we mean whether the additional of the variable, X_{3i}, to the model increases ESS (and hence R^2) "significantly" in relation to the RSS. This contribution is called **the incremental, or marginal** contribution of an additional variable.

To assess the incremental contribution of X_3 after allowing for the contribution of X_2, we form

ESS when we have x_2 & x_3 in the model (Q_3)

ESS when we have only x_2 in the model (Q_1)

$$\begin{aligned} \hat{F} &= \frac{Q_2/\text{df}}{Q_4/\text{df}} \\ &= \frac{(ESS_{new} - ESS_{old})/\text{number of new regressors}}{RSS_{new}/df(\text{=n-number of parameters in the new model})} \end{aligned} \tag{8.2}$$

Under the normality assumption of u_i and CLRM assumptions, this F value follows the F distribution with 1 and n-number of parameters in the new model.

RSS in the new model (w/ x_2 & x_3)

Table 8.3: ANOVA Table To Assess Incremental Contribution of A Variable(s)

Source of variation	Sum of Square **SS**	**df**	Mean Sum of Square **MSS**
from model 1: ESS due to X_2 alone	$Q_1 = \hat{\alpha}_2{}^2 \sum x_2^2$	1	$\frac{Q_1}{1}$
ESS due to the addition of X_3	$Q_2 = Q_3 - Q_1$	1	$\frac{Q_2}{1}$
from model 2: ESS due to both X_2,X_3	$Q_3 = \hat{\beta}_2 \sum x_{2i} y_i + \hat{\beta}_3 \sum x_{3i} y_i$	2	$\frac{Q_3}{2}$
RSS	$Q_4 = \underbrace{Q_5}_{TSS} - \underbrace{Q_3}_{ESS' \text{model 2}}$	n-3	$\frac{Q_4}{n-3}$
TSS	$Q_5 = \sum y_i^2$	n-1	

As usual method, we can re write 8.2 in term of R^2 only. Thus the F ratio of 8.2 is equivalent to the following F ratio:

$$
\begin{aligned}
\hat{F} &= \frac{R^2_{new} - R^2_{old}/\text{df}}{(1 - R^2_{new})/\text{df}} \\
&= \frac{(R^2_{new} - R^2_{old})/\text{number of new regressors}}{1 - R^2_{new}/df(\text{=n-number of parameters in the new model})}
\end{aligned}
\tag{8.3}
$$

This F ratio follows the F distribution with 1 and n-number of parameters in the new model.

Example

Consider the child mortality example. We considered the behavior of child mortality (CM) in relation to per capita GNP (PGNP). There we found that PGNP has a negative impact on CM, as one would expect. Now let us bring in female literacy as measured by the female literacy rate (FLR). A priori, we expect that FLR too will have a negative impact on CM. Our sample consists of 64 countries.

In model 1, we regressed child mortality (CM) on per capita GNP (PGNP) and female literacy rate (FLR).

Model 1:

$$
\begin{aligned}
\widehat{CM}_i &= \overset{\hat\beta_1}{263.6416} - \overset{\hat\beta_2}{0.0056}PGNP_i - \overset{\hat\beta_3}{2.2316}FLR_i \\
se &= (11.5932) \quad (0.0019) \qquad (0.2099) \quad R^2 = 0.7077
\end{aligned}
\tag{8.4}
$$

Now we extend this model to model 2 by including total fertility rate (TFR):

Model 2:

$$
\begin{aligned}
\widehat{CM}_i &= \overset{\hat\beta_1}{168.3067} - \overset{\hat\beta_2}{0.00555}GNP_i - \overset{\hat\beta_3}{1.7680}FLR_i + \overset{\hat\beta_4}{12.8686}TFR_i \\
se &= (32.8916) \quad (0.0018) \qquad (0.2480) \qquad (?) \quad R^2 = 0.7474
\end{aligned}
\tag{8.5}
$$

Questions

1.How would you choose between models 1 and 2? Which statistical test would you use to answer this question? Show the necessary calculations.
2. We have not given the standard error of the coefficient of TFR. Can you find it out? (Hint: Recall the relationship between the t and F distributions.

① $H_0: \beta_4 = 0$ (TFR should not be included)
$H_1: \beta_4 \neq 0$ (TFR should be included)

② $$\hat{F} = \frac{(R^2_{NEW} - R^2_{OLD})/df}{(1-R^2_{NEW})/\underbrace{df}_{(n-k)}} = \frac{(0.7474 - 0.7077)/1}{(1 - 0.7474)/(64-4)} = 9.4299$$

k = # parameters including intercept IN THE NEW MODEL

③ $$F_{CRITICAL} = F_{\underset{=0.05}{\alpha},\overset{N_1}{1},\overset{N_2}{60}} = 4.00$$

④ since $\hat{F} > F_{CRITICAL}$, then we may reject $H_0: \beta_4 = 0$.

Adding TFR significantly improves ESS and R^2.
So, TFR should be added into the model.

Q: $se(\hat{\beta}_4) = ?$

Recall that $\boxed{t^2 = F}$. Then $t = \sqrt{F} = \sqrt{9.4299} = 3.0781$

As $\boxed{t = \frac{\hat{\beta}_4}{se(\hat{\beta}_4)}}$, $se(\hat{\beta}_4) = \frac{\hat{\beta}_4}{t} = \frac{12.8686}{3.0781} = 4.1906$.

8.4 Testing the Equality of Two Regression Coefficients

Suppose we have the following model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \ldots + \beta_k X_{ki} + u_i$$

We would like to test the hypotheses:

$$H_0: \ \beta_3 = \beta_4 \ \text{ or } \ (\beta_3 - \beta_4) = 0$$
$$H_1: \ \beta_3 \neq \beta_4 \ \text{ or } \ (\beta_3 - \beta_4) \neq 0$$

Under the classical assumptions, it can be shown that:

$$t = \frac{(\hat{\beta}_3 - \hat{\beta}_4) - (\beta_3 - \beta_4)}{se(\hat{\beta}_3 - \hat{\beta}_4)}$$

where the t follows the t distribution with (n-k) df because the above equation is a k-variable model, where k is the total number of parameters estimated, including the constant term.

The $se(\hat{\beta}_3 - \hat{\beta}_4)$ is calculated from the following formula:

$$se(\hat{\beta}_3 - \hat{\beta}_4) = \sqrt{var(\hat{\beta}_3) + var\hat{\beta}_4 - 2cov(\hat{\beta}_3, \hat{\beta}_4)}$$

Example

among other things, you were asked to consider the following demand function for chicken:

$\hat{\beta}_2$ $\hat{\beta}_3$

$\hat{t} = \frac{\text{coeff}}{se}$

$$\begin{aligned}\widehat{\ln Y_t} &= 2.0328 + 0.4515 \ln X_{2t} - 0.3772 \ln X_{3t} \\ se &= (0.1162) \quad (0.0247) \qquad (0.0635) \quad R^2 = 0.9801\end{aligned} \tag{8.6}$$

where Y = per capita consumption of chicken, lb, X_2 = real disposable per capita income, \$, X_3 = real retail price of chicken per lb.

Question

For the above demand function, how would you test the hypothesis that the income elasticity is equal in value but opposite in sign to the price elasticity of demand? Show the necessary calculations. [Note: $\text{cov}(\hat{\beta}_2, \hat{\beta}_3) = -0.00142$. and the sample data = 23 observations]

① $H_0: \beta_2 + \beta_3 = 0$ OR $\beta_2 = -\beta_3$

$H_1: \beta_2 + \beta_3 \neq 0$ OR $\beta_2 \neq -\beta_3$

③ $$\hat{t} = \frac{(\hat{\beta}_2 + \hat{\beta}_3) - (\cancel{\beta_2 + \beta_3})^{0}}{\sqrt{var(\hat{\beta}_2) + var(\hat{\beta}_3) - 2cov(\hat{\beta}_2, \hat{\beta}_3)}}$$

$$= \frac{0.4515 + (-0.3772)}{\sqrt{(0.0247)^2 + (0.0635)^2 - 2(-0.00142)}}$$

$$\hat{t} = 0.8589$$

③ $t_{CRITICAL} = t_{\frac{\alpha}{2}, n-k} = t_{\frac{0.05}{2}, 23-3}$

$= t_{0.0025, 20} = 2.086$

④ since $\hat{t} = 0.8589 < t_{CRITICAL} = 2.086$,
then we fail to reject $H_0: \beta_2 + \beta_3 = 0$ OR $\beta_2 = -\beta_3$
at 95% confidence level.

Meaning : Price elasticity of demand AND Income elasticity of demand are equal in magnitudes but differ in signs.

8.5 Restricted Least Squares: Testing Linear Equality Restriction

In economic theories, the coefficients in a regression model need to satisfy some linear equality restrictions, For example, in microeconomics, consider the Cobb-Douglas production function:

$$Y_i = \beta_1 X_{2i}^{\beta_2} X_{3i}^{\beta_3} e^{u_i}$$

where Y=output, X_2 = labor input, and X_3=capital input. We can transform the above equation to be the log form as:

$$\ln Y_i = \beta_0 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i$$

where $\beta_0 = \ln \beta_1$

Now, if there are the constant returns to scale, economic theory would suggest that

$$\ln Y_i = \beta_0 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i$$

where $\beta_0 = \ln \beta_1$

Now, if there are the constant returns to scale, economic theory would suggest that

$$\boxed{\beta_2 + \beta_3 = 1}$$

$\% \Delta Q = \% \Delta L = \% \Delta K$

which is an example of a linear equality restriction.
In order to test the above linear equality restriction, we can follow two approaches which are:

[1]. The t-test approach

$\beta_2 = \frac{\% \Delta Y}{\% \Delta L}$ = labor-output elasticity

[2]. The F-test approach: Restricted Least Squares.

$\beta_3 = \frac{\% \Delta Y}{\% \Delta K}$ = capital-output elasticity

First Approach: The t-Test
A test of the hypothesis or restriction can be conducted by the t-test:

$$t = \frac{(\hat{\beta}_2 + \hat{\beta}_3) - (\beta_2 + \beta_3)}{se(\hat{\beta}_2 + \hat{\beta}_3)}$$

where the t follows the t distribution with (n-k) df for a k-variable model, where k is the total number of parameters estimated, including the constant term. In this case, df=n-3.

The $se(\hat{\beta}_2 + \hat{\beta}_3)$ is calculated from the following formula:

$$se(\hat{\beta}_2 + \hat{\beta}_3) = \sqrt{var(\hat{\beta}_2) + var\hat{\beta}_3 + 2cov(\hat{\beta}_2, \hat{\beta}_3)}$$

$\hat{t} = \frac{\text{coeff}}{se}$

Example

Consider the Cobb-Douglas production function to the Mexican economy (1955-1974: n=20):

$$
\begin{aligned}
\widehat{\ln GDP_t} &= -1.6524 + \overset{\hat{\beta}_2}{0.3397} \ln Labor_t + \overset{\hat{\beta}_3}{0.8460} \ln Capital_t \\
t &= (-2.7259) \quad (1.8295) \qquad (9.0625) \quad R^2 = 0.9951 \quad RSS_{UR} = 0.0136
\end{aligned} \tag{8.7}
$$

where GDP = Real GDP, Millions of 1960 pesos, *Labor* = Employment, Thousands of People, *Capital* = Fixed Capital, Millions of 1960 pesos.

Question

As you can see, the output/labor elasticity is about 0.34 and the output/capital elasticity is about 0.85. If we add these coefficients, we obtain 1.19, suggesting that perhaps the Mexican economy during the stated time period was experiencing increasing returns to scale. However, we do not know if 1.19 is statistically different from 1.

Therefore, we have to test this linear equality restriction.

$H_0: \beta_2 + \beta_3 = 1$ (CRS)

$H_1: \beta_2 + \beta_3 \neq 1$ (NOT CRS)

- COMPUTE $\hat{t}$:

$$\hat{t} = \frac{(\hat{\beta}_2 + \hat{\beta}_3) - (\beta_2 + \beta_3)}{\sqrt{var(\hat{\beta}_2) + var(\hat{\beta}_3) \pm 2\,cov(\beta_2 + \beta_3)}}$$

$$= \frac{(0.3397 + 0.8460) - 1}{\sqrt{(0.18567)^2 + (0.09335) + 2(-0.3319)}}$$

$$= 2.0499$$

- Find $t_{critical}$:

$$t_{0.025,\ \underset{n}{20} - \underset{k}{3}} = 2.110$$

- compare $\hat{t}$ with $t_{critical}$: since $\hat{t} = 2.0499 < t_{critical} = 2.110$, then we fail to reject the $H_0: \beta_2 + \beta_3 = 1$.

Therefore, the production function exhibits constant returns to scale: $\%\Delta K = \%\Delta L = \%\Delta Q$

8.6 The F-Test Approach: Restricted Least Squares

From the Cobb-Douglas production function:

$$\ln Y_i = \beta_0 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i \tag{8.8}$$

if there are the constant returns to scale, economic theory would suggest that

$$\beta_2 + \beta_3 = 1$$

We can rewrite it as:

$$\beta_2 = 1 - \beta_3$$

or

$$\beta_3 = 1 - \beta_2$$

Using either of these equalities, we can eliminate one of the β coefficients. Therefore, we can rewrite the Cobb-Douglas production function as:

Let's substitute $\beta_2 = 1 - \beta_3$ into (8.8):

$$\ln Y_i = \beta_0 + (1-\beta_3)\ln X_{2i} + \beta_3 \ln X_{3i} + u_i$$

$$= \beta_0 + \ln X_{2i} + \beta_3(\ln X_{3i} - \ln X_{2i}) + u_i$$

$$\ln Y_i - \ln X_{2i} = \beta_0 + \beta_3(\ln X_{3i} - \ln X_{2i}) + u_i$$

$$\ln(Y_i/X_{2i}) = \beta_0 + \beta_3 \ln(X_{3i}/X_{2i}) + u_i \tag{8.9}$$

→ we call this as " RESTRICTED MODEL "

where $\frac{Y_i}{X_{2i}}$ = output/labor ratio
$\frac{X_{3i}}{X_{2i}}$ = capital labor ratio.

$$\ln\left(\frac{GDP}{Labor}\right) = \beta_0 + \beta_3 \ln\left(\frac{CAPITAL}{LABOR}\right) + u_i$$

It should be noted that:
8.8 is known as **unrestricted Least Squares (URLS)**
8.9 is known as **restricted Least Squares (RLS)**

We can compare the unrestricted and restricted least-squares regressions by applying the F-test as follows:

$\sum \hat{u}_{i\,UR}^2$ [redacted]=RSS of the unrestricted regression 8.8

$\sum \hat{u}_{i\,R}^2$ [redacted] = RSS of the restricted regression 8.9

m=number of linear restrictions (in this example, we have 1 restriction)

k= number of parameters in the unrestricted regression

n= number of observations

Then, we have

$$
\begin{aligned}
\hat{F} &= \frac{(RSS_R - RSS_{UR})/m}{RSS_{UR}/(n-k)} \\
&= \frac{(\sum \hat{U}_R^2 - \sum \hat{U}_{UR}^2)/m}{\sum \hat{U}_{UR}^2/(n-k)}
\end{aligned}
\tag{8.10}
$$

follows the F-distribution with m, (n-k) df.

We can also rewrite the F-test in terms of R^2 as follows:

$$
\hat{F} = \frac{R_{UR}^2 - R_R^2/\mathrm{m}}{(1-R_{UR}^2)/\mathrm{n\text{-}k}}
\tag{8.11}
$$

Example

Consider the Cobb-Douglas production function to the Mexican economy(1955-1974: n=20):

$\hat{\beta}_2$ $\hat{\beta}_3$

$$\begin{aligned} \ln \widehat{GDP}_t &= -1.6524 + 0.3397 \ln Labor_t + 0.8460 \ln Capital_t \\ t &= (-2.7259) \quad (1.8295) \qquad (9.0625) \quad R^2 = 0.9951 \quad RSS_{UR} = 0.0136 \end{aligned} \tag{8.12}$$

UR

where GDP = Real GDP, Millions of 1960 pesos, *Labor* = Employment, Thousands of People, *Capital* = Fixed Capital, Millions of 1960 pesos.

$\beta_2 + \beta_3 = 1$

The restriction of constant return to scale, which gives the following regression:

$$\begin{aligned} \ln(\widehat{GDP/Labor})_t &= -0.4947 + 1.0153 \ln(Capital/Labor)_t \\ t &= (-4.0612) \quad (28.1056) \quad R_R^2 = 0.9777 \quad RSS_R = 0.0166 \end{aligned} \tag{8.13}$$

R

① SET HYPOTHESES : $H_0: \beta_2 + \beta_3 = 1$

$H_1: \beta_2 + \beta_3 \neq 1$

② COMPUTE $\hat{F}$: $\hat{F} = \dfrac{(RSS_R - RSS_{UR})/m}{RSS_{UR}/(n-k)}$

$= \dfrac{(0.0166 - 0.0136)/1}{0.0136/(20-3)}$

$= 3.75$

③ FIND CRITICAL F : $F_{\underset{\alpha}{0.05},\, \underset{m}{1},\, \underset{n-k}{17}} = 4.45.$

④ compare $\hat{F}$ AND $F_{CRITICAL}$: Since $\hat{F} = 3.75 < F_{CRITICAL} = 4.45$, then we cannot reject $H_0: \beta_2 + \beta_3 = 1$ at 95% confidence level.

Therefore, Mexican Economy follows the Constant Returns to Scale (CRS). #

STABILITY TEST OR STRUCTURAL BREAK TEST

8.7 Testing for Structural or Parameter Stability of Regression Models: The Chow Test

Sometime when we estimate the regression model, it may happen that there is a **Structural Change** in the relationship between the regressand Y and the regressors X's, especially the model involving time series data. The structural change may be due to the external forces (i.e the financial crisis of 2007-2008) or due to policy changes (such as the switch from a fixed exchange rate system to a flexible exchange rate system in 1997).

The question is **"How do we figure out that there is a structural change in our sample data?"**

To answer this question, consider the following example.

Based on the sample data, we found out that in 1982 the United State suffers its worst peacetime regression. This event might disturb the relationship between savings and DPI.

To see this effect, we can divide our sample data into two time periods: 1970-1981 (Pre-1982 crisis) and 1982-1995 (Post-1982 crisis).

Therefore we have three possible regressions:

Time period 1970-1981: $Y_t = \beta_1 + \beta_2 X_t + u_1 t$ where $n_1 = 12$

Time period 1982-1995: $Y_t = \gamma_1 + \gamma_2 X_t + u_2 t$ where $n_2 = 14$

Time period 1970-1995: $Y_t = \alpha_1 + \alpha_2 X_t + u_t$ where $n = n_1 + n_2 = 26$

For our sample data, we can get the following results:

Time period 1970-1981:

$$\begin{aligned} \hat{Y}_t &= 1.0161 + 0.0803 X_t \\ t &= (0.00873) \quad (9.6015) \end{aligned} \tag{8.14}$$

$R^2 = 0.9021 \quad RSS_1 = 1785.032 \quad df = 10$

$\Sigma \hat{u}_i^2$ for Pre-Crisis

$n-k = 12-2 = 10$

Time period 1982-1995:

$$\begin{aligned} \hat{Y}_t &= 153.4947 + 0.0148 X_t \\ t &= (4.6922) \quad (1.7707) \end{aligned} \tag{8.15}$$

$R^2 = 0.2971 \quad RSS_2 = 10,005.22 \quad df = 12$

$\Sigma \hat{u}_i^2$ for Post Crisis

$n-k = 14-2 = 12$

Time period 1970-1995:

$$\begin{aligned} \hat{Y}_t &= 62.4226 + 0.0376 X_t \\ t &= (4.8917) \quad (8.8937) \end{aligned} \tag{8.16}$$

$R^2 = 0.7672 \quad RSS_3 = 23,248.30 \quad df = 24$

$\Sigma \hat{u}_i^2$ for the whole period.

$n-k = 26-2 = 24$

We can apply **the Chow test** to investigate the structural changes that may be caused by differences in the intercept or the slope coefficient or both.

The chow test assumes that:

[1] $u_{1t} \sim N(0, \sigma^2)$ and $u_{2t} \sim N(0, \sigma^2)$

[2] The two error terms u_{1t} and u_{2t} are independently distributed.

Chow Test

H_0: There is no structural change in the model

H_1: There is structural change in the model

Then, we need to construct the F-ratio:

$$\hat{F} = \frac{(RSS_R - RSS_{UR})/k}{RSS_{UR}/(n_1 + n_2 - 2k)} \tag{8.17}$$

where the F ratio follows the F distribution with k and $(n_1 + n_2 - 2k)$ df in the numerator and denominator, respectively.

We do not reject the null hypothesis of parameter stability (i.e no structural change) if the computed F value does not exceed the critical value F value obtained from the F table.

If $\hat{F} < F_{critical}$, we fail to reject H_0

Chow test

① H_0: No Structural Change
H_1: Structural Change exists.

② Compute $\hat{F}$: $\hat{F} = \frac{(RSS_R - RSS_{UR})/k}{RSS_{UR}/(n_1 + n_2 - 2k)}$

$RSS_R = RSS_3 = 23,248.30$

$RSS_{UR} = RSS_1 + RSS_2 = 1785.032 + 10,005.22 = 11,790.252$

$n_1 + n_2 - 2k = 12 + 14 - 2 \cdot 2 = 22$

$\hat{F} = \frac{(23,248.30 - 11790.252)/2}{11790.252/22} = 10.69$

③ 1970 – 1981 ($n_1 = 12$) UR
② 1982 – 1995 ($n_2 = 14$) UR
③ 1970 – 1995 ($n = 26$) R

① ②
1970 1982 1995
③

RESTRICTED IN THE SENSE THAT TWO PERIODS' BEHAVIOR WERE "COMBINED" IN ONE REGRESSION

③ FIND CRITICAL F: $F_{0.01, 2, 22} = 5.72$

④ COMPARE $\hat{F}$ and CRITICAL F: Since $\hat{F} = 10.69 > F_{critical} = 5.72$, then we may reject H_0: No Structural change.

The Saving-Income Relationship was disturbed by 1982 CRISIS.