EC431: Topic 4. Capital Asset Pricing Model and Arbitrage Pricing Theory
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Copeland, Thomas E. and J. Fred Weston, Financial Theory and Corporate Policy (4th ed), Addison-Wesley(2005), HG4011 .C622: Ch6 (pp 147-157)

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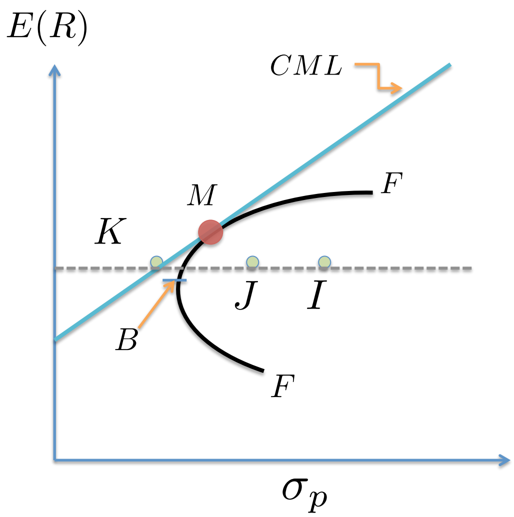
1 Portfolio Diversification and Individual Asset Risk



- Should "variance" be a good measurement of "risk" ?
- Asset I, J are .... the capital market line
- People do not hold asset I, J separately
- $E(R)$, σ^2 of asset I, J $\nrightarrow$ rate of returns the market will require from asset I, J
- Riskiness of asset I, J $\rightarrow$ their constribution in the asset portfolio (covariance risk)
- Equilibrium: CAPM , APT

The Single-Price Law of Securities : All securities or combinations of securities that have the same joint distributions of return will have the same price in equilibrium.

2 Should "variance" be a good measurement of "risk"?

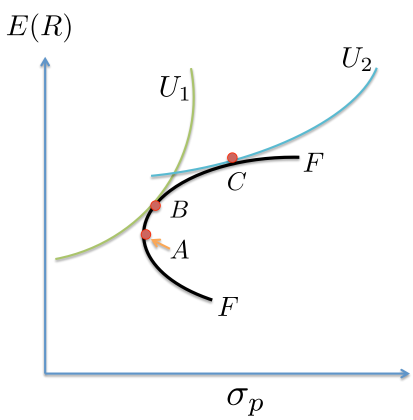
- People do not hold any single asset separately.
- For example : a share in a umbrella company. The share price varies depending on the weather condition (weather risk)
- very high $\sigma \Rightarrow$ very high required rate of return
- people do not hold a share in an umbrella company separately. They hold a market portfolio, which includes every asset in the economy.
- They hold shares in a sunblock manufacturer also.
- Holding both shares in a sunblock company and shares in an umbrella company $\Rightarrow \sigma_p \downarrow$
- The porfolio is doing well in all kind of weather (sunny, rainy)
- Therefore, even though the share of an umbrella company may have very high variance. They do not need compensation for weather risk.
- As a result, people will not require very high rate of return to hold hold a share of an umbrella company.
- People consider how individual asset contributes in their asset portfolio.
- Diversificable risk VS. Non-diversificable risk.
- People care only about "non-diversificable risk".
- (Explain in the details later)
- CML : $E(R_p) = R_f + \left(\frac{ER_m - R_f}{\sigma_m} \right) \sigma_p$
- Equilibrium relationship between $E(R_p)$ and σ_p for efficiently diversified portfolio (M, R_f)
- **NOT** Equilibrium relationship between required rate of return and risk for any other asset portfolio.

3 The CAPM

3.1 The CAPM : Assumptions

1. Investors are risk-averse, maximise expected utility of their wealth
2. Investors are price-takers, homogeneous expectations about expected asset returns, variance and covariance
3. There exists a risk-free asset such that investors may borrow or lend unlimitted amount at a risk-free rate
4. The quantities of assets are fixed. All assets are marketable and perfectly divisable
5. Asset market are frictionless, no information cost, no transaction cost
6. no market imperfections such as taxes, regulations, or restrictions on short selling

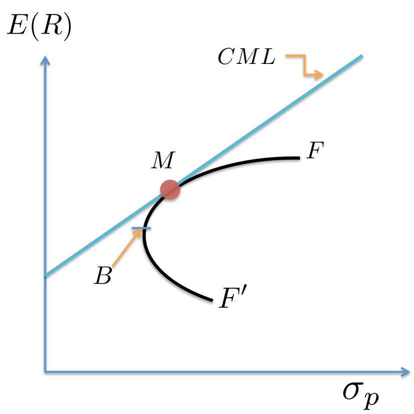
3.2 The CAPM : The Efficiency of the Market Porfolio



- Homogeneous beliefs $\rightarrow$ same efficient frontier
- Even without a risk-free asset, individuals will choose efficient portfolios regardless of their degree of risk aversion
- Market portfolio $\rightarrow$ consisting of all assets, held in proportion equals to its share in the market)
- The market is the sum of all individual holdings

- **The market portfolio must lie on the efficient frontier**

3.3 Derivation of the CAPM



- Homogeneouse beliefs $\rightarrow$ same efficient frontier, same CML
- In equilibrium: Demand = Supply
- The prices must adjust until all assets are held
- All individuals hold a combination of a risk-free asset and the portfolio M
- All investors hold the same risky portfolio such that the portfolio weights
- M must be a market portfolio

Theorem:

1. In equilibrium, all investors hold the market portfolio. The proportion of each asset in the market portfolio is

   $$w_i = \frac{\text{market value of the individual asset } i}{\text{market value of all assets}}.$$

2. The equilibrium expected return on the ith security is

   $$ER_i = R_f + (ER_M - R_f)\frac{\sigma_{im}}{\sigma_m^2}$$

Proof will come later.

3.4 Proof of 1. In equilibrium, all investors hold the market portfolio (and risk free asset).

- In equilibrium, all investors hold the market portfolio. The proportion of each asset is equal to proportion in the market portfolio.
- Example : Suppose there are only two risky assets in the market, Stock A and Stock B.

	Price	Number of Stocks	Value
Stock A	$2	30	
Stock B	$1	40	
Market	-	-	

- Market Portfolio consists of .... % of wealth invested in Stock A and ....% of wealth invested in Stock B.
- All investors hold the same "risky portfolio".
- At equilibrium all investors hold the "market portfolio".
- All investors hold the same ratio of Stock A and Stock B.
- On the demand side, the market demand is simply the sum of individual holdings.
- Therefore, total market demand contains Stock A and Stock B at the same weight as the market portfolio.
- At equilibrium price, aggregate demand = aggregate supply and all individuals hold the market portfolio.
- Homogenous belief $\Rightarrow$ All investors hold the same "risky portfolio".
- What will happen if all investors does not hold "the market portfolio".
- Suppose all investors hold a risky portfolio consisting 50% of Stock A and 50% of Stock B.
- Then, the market demand is not equal to the maket supply.
- This means that these prices are not equlibrium prices.
- The prices of all assets must adjust until all are held by investors.
- (Note that this is not a "formal" proof of the theory. A formal proof can be found in an advanced textbook)

3.5 Proof of 2. CAPM equation

- The equilibrium expected return on the ith security is
  $ER_i = R_f + (ER_M - R_f)\frac{\sigma_{im}}{\sigma_m^2}$.

- Consider a portfolio consisiting of a % invested in risky asset i and $(1-a)$% in the market portfolio (a will be equal to 0 later)

$$E(R_p) = aER_i + (1-a)ER_m$$

$$\sigma^2(R_p) = a^2\sigma_i^2 + 2a(1-a)\sigma_{im} + (1-a)^2\sigma_m^2$$

$$\sigma(R_p) = \left[a^2\sigma_i^2 + 2a(1-a)\sigma_{im} + (1-a)^2\sigma_m^2\right]^{\frac{1}{2}}$$

where

σ_i^2 =the variance of the risky asset i, σ_m^2 =the variance of the market portfolio, σ_{im} =the covariance between the risky asset i and the market portfolio

- $\frac{\partial E(R_p)}{\partial a} =$

- $\frac{\partial \sigma(R_p)}{\partial a} =$

- In equilibrium, $a = 0$. The slope of the efficient portfolio at point M : $\left.\dfrac{\partial E(R_p)/\partial a}{\partial \sigma_p/\partial a}\right|_{a=0}$

- $\dfrac{\partial \sigma(R_p)}{\partial a} = \dfrac{\sigma_{im} - \sigma_m^2}{\sigma_m}$

- $\dfrac{\partial E(R_p)/\partial a}{\partial \sigma_p/\partial a} =$

- Recall CML line : $E(R_p) = R_f + \left(\dfrac{E(R_M) - R_f}{\sigma_M}\right)\sigma_p$

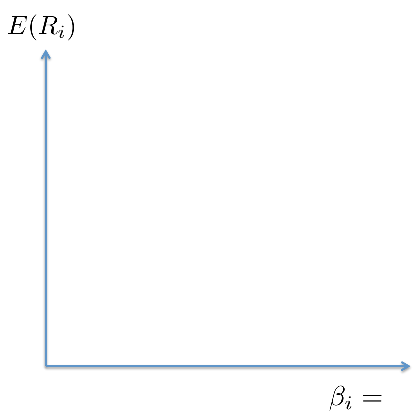
- The tangency portfolio is the market portfolio; the slopes must be equal

- Rearrange, solve for $E(R_i)$

This equation is known as *the capital asset pricing model*, CAPM.

- It shows the relationship between the required rare of return on any asset and the quantity of risk (β)
- $\beta_i = \dfrac{\sigma_{im}}{\sigma_m^2} =$
- $\beta_m =$
- High beta shares have $\beta > 1$ are called aggressive shares. When the return on the market goes up, ie. in a bull market, return on high beta shares will go up faster than the market.
- Low Beta shares have$\beta < 1$ are called defensive shares. When the return on the market goes down ie. in a bear market, return on the low beta shares will fall less than the market.

3.6 The CAPM can be shown graphically.



- *Security Market Line* (SML)
- CML : the relationship between expected return and "total" risk for efficiently diversified portfolios
- SML : "equilibrium relationship" between expected return and β - "*Systematic Risk*" (explain later)

3.7 Properties of the CAPM

3.7.1 Covariance risk

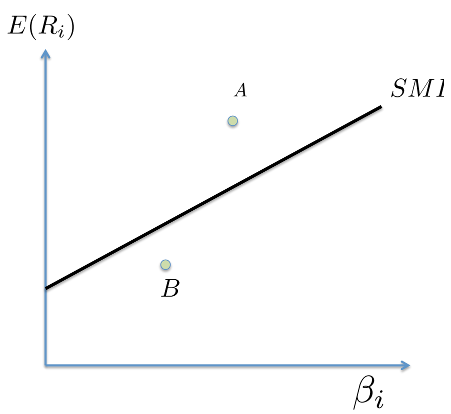
- The only risk that investors will pay a premium to avoid is covariance risk : $\beta_i = \dfrac{COV(R_i, R_m)}{VAR(R_m)}$
- Investors can always diversify away all risk except the covariance risk
- The total risk = systematic risk(aggregate risk) + unsystematic risk (ideosyncratic risk)

   1. "a general component representing that portion in the variability of a stock's total returns that is directly associated with overall movements in general economic (or stock market) activity."
   2. "a specific (issuer) component, representing that portion in the variability of the stock's return that is not related to the variability in general economic (market) activity.

- Examples for unsystematic risk are business risk (in wrong business), financial risk (too much debt/equity)
- "Virtally all securities have some systematic risk, whether bonds or stocks, because systematic risk directly encompasses interest rate risk, market risk and inflation risk."
- "As more securities are added, the non-systematic risk becomes smaller, and smaller, and the total risk for the portfolio approaches its systematic risk."
- Diversification cannot reduce systematic risk.
- Aggregate risk cannot be diversified away.
- "By investing in a diversified portfolio of shares the impact of any single company's unique risk can be reduced. This works very simply because the returns from individual company do not move entirely together; i.e. they are not perfectly correlated. Consequently external influences, which dramatically affect the price of share in the portfolio, may have almost no influence, or even an opposite impact, on the price of another share in the same portfolio. Thus, provided investors the spread their total investment across a sufficient number of shares the risk of their overall portfolio should be reduced below the risk of the individual shares comprising the portfolio."
- "However, no level of diversification can reduce or get rid of the second component of risk. This is known as market or systematic risk and refers to the intrinsic risk of investing in the stock market given thath companies are subject to greater volatility than many other financial investments such as government bonds. There are external business environmental factors, such as worldwide recessions and economic boom, which affect all businesses, but they do not affect all businesses to the same extent."
- "The relative sensitivity of a particular company's return to changes in the returns of the total stock market is called "*beta*" of the company. If the company is twice sensitive as the stock market, it is said to have a beta of 2; i.e. when overall stock market returns increase by 5 %, the returns on this share rise by 10%. Simiarly, a less sensitive company may have a beta of 0.75 which means that if the returns on stock market falls by 10%, this company returns will only fall by 7.5%."
- Mathematically; $(R_i - R_f) = b_i(R_m - R_f) + \epsilon_i$, OLS regression
- ϵ_i is a random variable; $E(\epsilon_i) = 0$, $COV(\epsilon_i, \epsilon_j) = 0$, $COV(\epsilon_i, R_m) = 0$
- $\sigma_i^2 =$

$$
\begin{aligned}
\sigma_i^2 &= \sigma_{\epsilon_i}^2 + \beta_i \sigma_{im} \\
&= \text{non-systematic risk + systematic risk} \\
&= \text{Specific Risk +Market Risk} \\
& \quad \text{ideasyncratic risk + aggregate risk} \\
&= \text{diversificable risk + non-diversificable risk}
\end{aligned}
$$

3.7.2 In equilibrium, every asset falls on SML



- **In equilibrium, every asset must be priced so that its required rate of return falls on SML**
- Asset A is .... (undervalued/overvalued?)
  - investors would .... (buy or sell ? more A)
  - then, the price of asset A will .... (rise/fall?)
  - its required rate of return will .... (rise/fall?)
- Asset B is ....
  - investors would ....
  - then, the price of asset B will ....
  - its required rate of return will ....

$$Price = \frac{Payoff}{1 + \text{required rate of return}}$$

3.8 CAPM : Summary

- **CML** : $E(R_p) = R_f + \left(\frac{E(R_m) - R_f}{\sigma_m}\right)\sigma_p$

- **CML** shows the relationship between expected return and "total" risk for efficiently diversified portfolios. Investors hold portfolios along the CML. Their market required rate of returns are determined by the relationship between the market required rate of return and standard deviation, as shown by CML equation.

- Portfolios which do not lie on the CML are inefficient. Though we know the variance and the rate of return on those portfolios, we cannot be sure at what rate of return that the market will require from them in order to hold them in equilibrium. The variance may not be the correct measure of riskiness for an individual asset.

- Hence, we cannot use our knowledge of the mean and the variance of an individual asset to determine its market required rate of return.

- We derive the CAPM by equating the slope of CML to the slope of efficient frontier at point M.

- **SML** : $E(R_i) = R_f + \left((E(R_m) - R_f\right)\beta_i$; $\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{COV(R_m, R_i)}{VAR(R_m)}$

- **SML** : "equilibrium relationship" between expected return and β - "*Systematic Risk*".

- Only the portion of total variance that is correlated with the economy(covariance risk) is relevant. Any portion of total risk that is not correlated with the economy is irrelevant and can be avoided at zero cost through diversification. We need the information on how the rate of return on an asset is correlated with the economy(covariance risk, systematic risk, β) to determine the market required rate of return on an asset. All assets will fall on SML in equilibrium.

- According to the CAPM, $R_i = R_f + \beta_i(R_m - R_f) + \epsilon_i$, $ER_i = R_f + \beta_i\left(ER_m - R_f\right)$

- $R_i = a_i + \beta_i R_m + \epsilon_i$

- $\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{\epsilon_i}^2$: Total risk = systematic risk + unsystematic risk.

4 The Abitrage Pricing Theory

- CAPM: $R_i = (R_f - \beta_i R_f) + \beta_i R_m + \epsilon_i$, $R_i = \beta_0 + \beta_i R_m + \epsilon_i$
  - only one source of systematic risk, unsystematic risk can be eliminated through diversification
- The APT: several sources of risk that cannot be eliminated through diversification
- The sources of risk : inflation, aggregate output, .. etc.
- APT: $R_i = \beta_0^i + \beta_1^i(\text{Return on Factor 1}) + \beta_2^i(\text{Return on Factor 2}) + ... + \beta_k^i(\text{Return on Factor k}) + \epsilon^i$
- Assets with the same values of β_j for all factors j must have the same rate of returns.
- If there is only one factor and that factor is R_m, the APT is the same as the CAPM.
- The APT is more general than the CAPM

4.1 Arbitrage opportunity and Equilibrium

- Arbitrage : an arbitrage opportunity arises when an investor can construct a zero investment portfolio that will yield a sure profit (risk-free)
- Example :
-

	S_1	S_2	$Price$
A_1	1	0	0.3
A_2	0	1	0.1

- If there exists a security C pay 2$ when S_1 occurs and it is priced at $ 0.8, we can construct an arbitrage portfolio; selling at a high price and buying at a low price.
- In equilibrium, no arbitrage portfolio exists.
- Law of one price :

4.2 No Arbitrage Equilibrium : Proof backward

- Example : $E(R_a) = 0.08 + 0.6F_1$, $E(R_b) = 0.02 - 0.2F_1$.
  - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has no risk ($\beta_1 = ...$).
  - If the risk free rate is equal to 0.01 (you can lend or borrow at 1% interest rate), can you make an arbitrage profit?

- At equilibrium, there must be no abitrage opportunity.
- From the last example, $E(R_a) = 0.08 + 0.6F_1$, $E(R_b) = 0.02 - 0.2F_1$.
  - Suppose there is asset C, $E(R_C) = 0.10 + 0.6F_1$.
  - Can you make an arbitrage profit?
  - Buy low and sell high.
  - Buy .... and short-sell ....
  - get a profit of ....=0.02 = 2%, regardless of the value of F_1.
- Short-sell : The sale of shares not owned by the investor but borrowed through a broker and later repurchased to preplace the loan. Profit is earned if the initial sale is at higher price than the repurchase price.
- $E(R_i) = \beta_0^i + \beta_1^i F_1 + \beta_2^i F_2 + ... + \beta_k^i F_K$
- Assets with the same values of β_j for all factors j must have the same rate of returns.

- Example : $E(R_a) = 0.08 + 0.6F_1$, $E(R_b) = 0.02 - 0.2F_1$.
  - R_f = the rate of return of the portfolio which has $\beta_1 = 0$.
  - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has $\beta_1 = 1$.
  - $R_m = 0.11 + F_1$
  - Determine the rate of returns of portiolio i which has 0, 0.6, 0.2, 1, 0.5, 2, 0.04, 0.08... etc .
- $E(R_i) = R_f + (E(R_m) - R_f)\beta_1^i$
- $E(R_i) =$ ....+ ....

- $\beta = 0.6 \Rightarrow E(R_i) = 0.035 + 0.6(0.11 + F_1 - 0.035) = 0.08 + 0.6F_1$
- $\beta = -0.2 \Rightarrow E(R_i) = 0.035 - 0.2(0.11 + F_1 - 0.035) = 0.02 - 0.2F_1$

- CAPM :
  $R_i = R_f + \beta$(Market Risk Premium)
  $R_i = R_f + \beta(E(R_{m)} - R_f)$

- APT :
  $E(R_i) = \beta_0^i + \beta_1^i$(Risk premium for Factor 1)$+ \beta_2^i$(~ for Factor 2)$+ ... + \beta_k^i$(~for for Factor k)

- $E(R_i) = R_f + (\lambda_1 - R_f)\beta_1^i + (\lambda_2 - R_f)\beta_2^i + ... + (\lambda_k - R_f)\beta_k^i$;

  λ_j = the expected rate of return on portfolio with unit sensitivity to the j the factor and zero sensitivity to all other factors.
- At equilibrium, there is no arbitrage opportunity.
- Assets with the same values of β_j for all factors j must have the same rate of returns.

- Examples: 2 Factors

  $E(R_a) = 0.10F_1 - 0.5F_2$

  $E(R_b) = 0.08 + 2F_1 + F_2$

  $E(R_c) = 0.05 + 0.5F_1 + 0.5F_2$

  $E(R_p) = w_a E(R_a) + w_b E(R_b) + w_c E(R_c)$

  - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has no risk

    $0.10w_a + 2w_b + 0.5w_c = 0$

    $-0.5w_a + 1w_b + 0.5w_c = 0$

    $w_a = \frac{5}{13},\ w_b = -\frac{3}{13},\ w_c = \frac{11}{13}$
  - Determine the rate of return on portfolio which has $\beta_1 = 1$ and $\beta_2 = 0$.
  - Determine the rate of return on portfolio which has $\beta_1 = 0$ and $\beta_2 = 1$

- Example : $E(R_i) = R_f + (\lambda_1 - R_f)\beta_1^i + (\lambda_2 - R_f)\beta_2^i$; λ_j = the expected rate of return on portfolio which has $\beta_j = 1$ and other betas equal to zero.

  $$\begin{aligned} ER^A &= 0.035 \\ ER^B &= 0.08 + F_2 \\ ER^C &= 0.11 + F_1 \end{aligned}$$

  - Determine the portfolio weights you need to place on a, b and c in order to construct a portfolio which has no risk
  - Determine the rate of return on portfolio which has $\beta_1 = 1$ and $\beta_2 = 0$.
  - Determine the rate of return on portfolio which has $\beta_1 = 0$ and $\beta_2 = 1$
  - Write down the general form for APT model