Solution Part I: Exercise for Assignment 5

1. Let $A = \{1, 2, 3\}$ and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{P}(A)$ be the set of all subsets of the set A. Define a relation r from $\mathcal{P}(A)$ to $\mathbb{Z}$ as

$$r = \{(x, y) \in \mathcal{P}(A) \times \mathbb{Z} | y = \text{ the number of elements in } x\}.$$

   (a) List all the elements in r.

   (b) Is r a function? If so, find the domain, co-domain, and range of r.

Solution:

   (a) The relation r is given by

$$r = \Big\{\{\emptyset, 0\}, \{\{1\}, 1\}, \{\{2\}, 1\}, \{\{3\}, 1\}, \{\{1, 2\}, 2\}, \{\{1, 3\}, 2\}, \{\{2, 3\}, 2\}, \{\{1, 2, 3\}, 3\}\Big\}.$$

   (b) Yes, r is a function because(i) every element in the domain is used and (ii) for each $x \in \mathcal{P}(A)$, there is a unique $y \in \mathbb{Z}$ so that $(x, y) \in r$, or if $(x, y_1), (x, y_2) \in r$ then $y_1 = y_2$ (i.e. each $x \in \mathcal{P}(A)$ gets mapped exactly once).
   The domain is $\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$.
   The co-domain is $\mathbb{Z} = \Big\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\Big\}$.
   The range is $\Big\{0, 1, 2, 3\Big\}$.

2. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define a function $f : X \to Y$ by $f = \{(1, a), (2, a), (3, c)\}$.

   (a) Find the domain of f, co-domain of f,and range of f.

   (b) What is the inverse image of a?

   (c) What is $f(2)$?

   (d) Draw the arrow diagram of f.

Solution:

   (a) The domain of f is $X = \{1, 2, 3\}$,
   co-domain of f is $Y = \{a, b, c, d\}$,and range of f is $\{a, c\}$.

   (b) The inverse image of a is $\{1, 2\}$ (because there are two elements with second component being a:$(1, a), (2, a)$).

   (c) Since $(2, a) \in f$, then $f(2) = a$

   (d) The arrow diagram of f:

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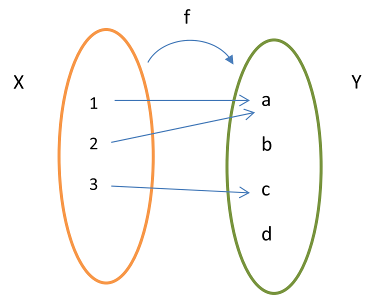

Figure 1: Problem 2(d)

3. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (2x + 1, \tfrac{1-y}{2}) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

(a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

Solution:

(a) Yes, H is one-to-one. To prove this, we will show that for (x_1, y_1) and (x_2, y_2) in the domain, if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$. Suppose $H(x_1, y_1) = H(x_2, y_2)$, Then

$$\begin{aligned} H(x_1, y_1) &= H(x_2, y_2) \\ \left(2x_1 + 1, \frac{1-y_1}{2}\right) &= \left(2x_2 + 1, \frac{1-y_2}{2}\right) \end{aligned}$$

and this is equivalent to $2x_1 + 1 = 2x_2 + 1$ and $\frac{1-y_1}{2} = \frac{1-y_2}{2}$, or

$$2x_1 + 1 = 2x_2 + 1 \quad \Leftrightarrow \quad 2x_1 = 2x_2 \quad \Leftrightarrow \quad x_1 = x_2$$

$$\frac{1-y_1}{2} = \frac{1-y_2}{2} \quad \Leftrightarrow \quad 1 - y_1 = 1 - y_2 \quad \Leftrightarrow \quad y_1 = y_2$$

i.e., $(x_1, y_1) = (x_2, y_2)$. That is, $H(x_1, y_1) = H(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$. ■

(b) Yes, H is onto. To prove this, we will show that for any (u, v) in the co-domain $\mathbb{R} \times \mathbb{R}$, there exists (x, y) from the domain $\mathbb{R} \times \mathbb{R}$ such that $H(x, y) = (u, v)$.
Suppose, temporarily that, there is (x, y) such that $H(x, y) = (u, v)$. Then,

$$\begin{aligned} H(x, y) &= (u, v) \\ (2x + 1, \frac{1-y}{2}) &= (u, v) \end{aligned}$$

i.e., we must have $2x + 1 = u \quad \Leftrightarrow \quad \boxed{x = \frac{u-1}{2}}$ and $\frac{1-y}{2} = v \quad \Leftrightarrow \quad \boxed{y = 1 - 2v}$.
Since $u, v \in \mathbb{R}$, then $x = \frac{u-1}{2}, y = 1 - 2v \in \mathbb{R}$.

That is, for *any* given (u,v) in the co-domain $\mathbb{R}\times\mathbb{R}$, we can find $(x,y) = \left(\frac{u-1}{2}, 1-2v\right) \in \mathbb{R}\times\mathbb{R}$ in the domain, such that

$$H(x,y) = H\left(\frac{u-1}{2}, 1-2v\right) = \left(2\frac{u-1}{2}+1, \frac{1-[1-2v]}{2}\right) = (u,v),$$

and hence H is onto. ■

(c) From (a) and (b), since H is both one-to-one and onto, then H bijective.
To find the inverse function, H^{-1}, of H, we recall from the definition

$$H^{-1}(u,v) = (x,y) \qquad \Leftrightarrow \qquad H(x,y) = (u,y).$$

Since we have from (b) that

$$H\left(\frac{u-1}{2}, 1-2v\right) = (u,v),$$

and hence the inverse function $H^{-1} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ is given by

$$H^{-1}(u,v) = \left(\frac{u-1}{2}, 1-2v\right).$$

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4. Let $f : (0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} |x|, & x \in (0,1] \\ x^2 & x \in (1,3] \\ 6+|x|, & x \in (3,\infty). \end{cases}$$

(a) Is f one-to-one? Prove or give a counterexample.
(b) Is f onto? Prove or give a counterexample.
(c) Is f bijective? If so, find the inverse function of f.

Solution:

First notice that since the values in the domain of f is positive so that $|x| = x$ for $x \in (0,\infty)$ and we can write f as

$$f(x) = \begin{cases} x, & x \in (0,1] \\ x^2 & x \in (1,3] \\ 6+x, & x \in (3,\infty). \end{cases}$$

(a) Is f one-to-one? Prove or give a counterexample.
To show that f is one-to-one, consider the definition (and the version of its contrapositive) . For any $x_1, x_2 \in X$,

$$\text{(I)} \quad \text{if} \quad f(x_1) = f(x_2), \quad \text{then} \qquad x_1 = x_2$$

and the equivalent definition from its contrapositive

$$\text{(II)} \quad \text{if} \quad x_1 \neq x_2 \quad \text{then} \qquad f(x_1) = f(x_2).$$

We will consider two main cases (with 3 sub-cases for each)
(i) If x_1 and x_2 are from the same subinterval, it is appropriate to use definition in (I) to show that, for any $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

- For $x_1, x_2 \in (0,1]$, $f(x_1) = x_1, f(x_2) = x_2$.
  So , if $f(x_1) = f(x_2)$, then we must have $x_1 = f(x_1) = f(x_2) = x_2$. I.e. $x_1 = x_2$.
- For $x_1, x_2 \in (1,3]$, $f(x_1) = x_1^2, f(x_2) = x_2^2$.
  So , if $f(x_1) = f(x_2)$, then we must have $x_1^2 = f(x_1) = f(x_2) = x_2^2$.
  Since $x_1, x_2 > 0$, then $x_1^2 = x_2^2$ implies $x_1 = x_2$.
- For $x_1, x_2 \in (3,\infty)$, $f(x_1) = x_1 + 6, f(x_2) = x_2 + 6$.
  So , if $f(x_1) = f(x_2)$, then

$$x_1 + 6 = x_2 + 6 \quad \Rightarrow \quad x_1 = x_2.$$

(ii) If x_1 and x_2 are from different subintervals $(0,1], (1,3], (3,\infty)$, f has different formulas and it is appropriate to use definition in (II):for any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

- For $x_1 \in (0,1]$ and $x_2 \in (1,3]$, we have $f(x_1) = x_1$ and $f(x_2) = x_2^2$.
  $x_1 \in (0,1] \Rightarrow f(x_1) = x_1 \in (0,1]$
  $x_2 \in (1,3] \Rightarrow f(x_2) = x_2^2 \in (1,9]$
  and hence $f(x_1) < f(x_2)$. That is, $x_1 \neq x_2 \quad \Rightarrow f(x_1) \quad \neq f(x_2)$.
- For $x_1 \in (0,1]$ and $x_2 \in (3,\infty)$, we have $f(x_1) = x_1$ and $f(x_2) = x_2 + 6$.
  $x_1 \in (0,1] \Rightarrow f(x_1) = x_1 \in (0,1]$
  $x_2 \in (3,\infty) \Rightarrow x_2 > 3 \Rightarrow x_2+6 > 3+6$ or $f(x_2) = x_2+6 > 9 \Rightarrow f(x_2) = x_2+6 \in (9,\infty)$
  and hence $f(x_1) < f(x_2)$. That is, $x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$.
- For $x_1 \in (1,3]$ and $x_2 \in (3,\infty)$, we have $f(x_1) = x_1^2$ and $f(x_2) = x_2 + 6$.
  $x_1 \in (1,3] \Rightarrow f(x_1) = x_1^2 \in (1,9]$
  $x_2 \in (3,\infty) \Rightarrow x_2 > 3 \Rightarrow x_2+6 > 3+6$ or $f(x_2) = x_2+6 > 9 \Rightarrow f(x_2) = x_2+6 \in (9,\infty)$
  and hence $f(x_1) < f(x_2)$. That is, $x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$.

Since cases (i) and (ii), together with their subcases, cover all possible values of x_1 and x_2 from the domain, we have that f is one-to-one. ■

(b) f is not onto on the co-domain $\mathbb{R}$. Since $f(x) > 0$ for all $x \in (0,\infty)$. So when we choose, for example, $x = -1$ from the codomain, we cannot find $x \in (0,\infty)$ such that $f(x) = -1$.

(c) Since f is not onto, then it is not bijective and there is no inverse function for f. ■

5. Find the largest sets D and S such that the function $f : D \to S$ defined by $f(x) = \frac{2}{x-1}$ is an onto function.

Solution: The largest domain is $D = \mathbb{R} - \{-1\}$
In order to be an onto function, consider $y = \frac{2}{x-1}$

$$x = \frac{2}{y} + 1.$$

Notice that when $y = 0$, we cannot find x in the domain such that $y = f(x)$ and therefore the largest set for S is $\mathbb{R} - \{0\}$. ■

6. Let $f : \mathbb{R} - \{1\} \to S$ be a *bijective* function defined by $f(x) = \frac{x+1}{x-1}$.

   (a) Determine the set S.
   (b) Determine the inverse function f^{-1}.

(c) Compute $f \circ f$ and $f \circ f^{-1}$.

Solution:

(a) Notice that, for $f(x) = \frac{x+1}{x-1}$, we have

$$y = \frac{x+1}{x-1} \quad \Leftrightarrow \quad y(x-1) = (x+1) \quad \Leftrightarrow \quad yx - x = 1 + y \quad \Leftrightarrow \quad x = \frac{1+y}{y-1}.$$

In order for f to be onto, we have to set $\boxed{S = \mathbb{R} - \{1\}}$.
Note that we can check that f is one-to-one, by letting x_1, x_2 be elements in the domain $\mathbb{R} - \{1\}$. Suppose $f(x_1) = f(x_2)$. Then

$$\begin{aligned} \frac{x_1+1}{x_1-1} &= \frac{x_2+1}{x_2-1} \\ (x_1+1)(x_2-1) &= (x_2+1)(x_1-1) \\ x_1x_2 + x_2 - x_1 - 1 &= x_1x_2 + x_1 - x_2 - 1 \\ x_2 - x_1 &= x_1 - x_2 \\ 2x_2 &= 2x_1 \\ x_2 &= x_1. \end{aligned}$$

(b) Since f is bijective (one-to-one and onto) we can find its inverse function. To determine the inverse function f^{-1}, we first notice from (a) that for $x = \frac{y+1}{y-1}$,

$$f\left(\frac{y+1}{y-1}\right) = \frac{\frac{y+1}{y-1} + 1}{\frac{y+1}{y-1} - 1} = \frac{\frac{y+1+y-1}{y-1}}{\frac{y+1-y+1}{y-1}} = \frac{2y}{2} = y.$$

Therefore, $f^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by

$$f^{-1}(y) = \frac{y+1}{y-1}.$$

Remark: $f = f^{-1}$.

(c) Compute $f \circ f$ and $f \circ f^{-1}$. First notice that, since the domain and the co-domain of both f and f^{-1} are the same, $f \circ f$ and $f \circ f^{-1}$ exist.
The function $f \circ f : \mathbb{R} - \{-1\} \to \mathbb{R} - \{-1\}$ is defined by

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{x+1}{x-1}\right) \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-x+1}{x-1}} = \frac{2x}{2} = x.$$

The function $f \circ f^{-1} : \mathbb{R} - \{-1\} \to \mathbb{R} - \{-1\}$ is defined by

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f\left(\frac{y+1}{y-1}\right) = \frac{\frac{y+1}{y-1} + 1}{\frac{y+1}{y-1} - 1} = \frac{\frac{y+1+y-1}{y-1}}{\frac{y+1-y+1}{y-1}} = \frac{2y}{2} = y.$$

Remark: $f \circ f = f \circ f^{-1}$ which is equal to the *identity function on* $\mathbb{R} - \{1\}$.

7. Graph the function $f : \mathbb{R} \to \mathbb{Z}^+ \cup \{0\}$, $f(x) = \lfloor x \rfloor - \lceil x \rceil$ on the closed interval $[-4, 4]$.

Solution:

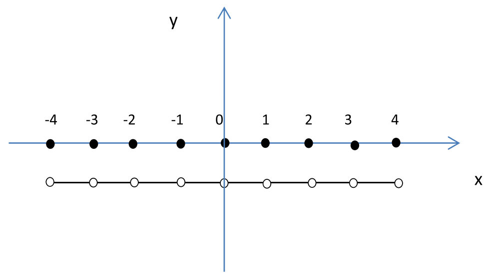

Figure 2: Problem 7

8. If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f : X \to Z$ is onto, must both f and g be onto? Prove or give a counterexample.

Solution: No, when $g \circ f : X \to Z$ is onto, it is not necessarily that both f and g have to be onto. Counterexample: Let $f : X \to Y$ and $g : Y \to Z$ be functions with $X = \{a, b, c\}$, $Y = \{1, 2, 3, 4\}$, $Z = \{r, s, t\}$. where

$$f = \{(a, 1), (b, 2), (c, 3)\}$$

and

$$g = \{(1, r), (2, s), (3, t), (4, t)\}.$$

Notice that f is not onto because it does not map any element in the domain to the element "4" in the co-domain Y. We have that $g \circ f : X \to Y$ defined by:

$$g \circ f = \{(a, r), (b, s), (c, t)\}$$

is onto (also shown in the diagram). Hence $g \circ f : X \to Z$ can be onto, even when f is not onto.

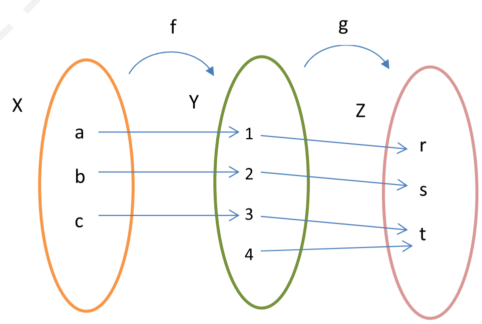

Figure 3: Problem 8

9. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1.$

   (b) $f(x) = x^5, \quad g(x) = x^{1/5}.$

   Solution: First notice that all domains and ranges of f and g for both (a) and (b) are the set of real numbers $\mathbb{R}$ and therefore it is possible to define $f \circ g$ and $g \circ f$.

   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1$. The function $f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by

   $$(f \circ g)(x) = f(g(x)) = \frac{x^3+1}{\sqrt{(x^3+1)^2+1}}$$

   and function $g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by

   $$(g \circ f)(x) = g(f(x)) = \left(\frac{x}{\sqrt{x^2+1}}\right)^3 + 1.$$

   So we have $(f \circ g)(2) = \frac{2^3+1}{\sqrt{(2^3+1)^2+1}} = \frac{9}{\sqrt{82}}$ and $(g \circ f)(2) = \left(\frac{2}{\sqrt{2^2+1}}\right)^3 + 1 = \frac{8+5\sqrt{5}}{5\sqrt{5}}$. Notice that since $(f \circ g)(2) \neq (g \circ f)(2)$, then $f \circ g \neq g \circ f$. ■

   (b) $f(x) = x^5, \quad g(x) = x^{1/5}$. The function $f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by

   $$(f \circ g)(x) = f(g(x)) = \left(x^{1/5}\right)^5 = x.$$

   and function $g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by

   $$(g \circ f)(x) = g(f(x)) = \left(x^5\right)^{1/5} = x.$$

   Notice that since $(f \circ g)(x) = (g \circ f)(x)$ for all $x \in \mathbb{R}$, then $f \circ g = g \circ f$. So we have $(f \circ g)(2) = (g \circ f)(2) = 2$ ■

10. Let f and g be functions defined by $f(x) = \sqrt{2-x}, x \geq 2$ and

   $$g(x) = \begin{cases} x^2 - 1, & x \in (-\infty, -1] \\ \frac{1}{x}, & x \in (-1, \infty). \end{cases}$$

   Find $g \circ f$ and its domain.

   Solution: Notice that f is defined only when $x = 2$ which gives $f(2) = 0$. Hence the range of f is $\{0\}$. From the formula of g, the domain of g is $\mathbb{R} - \{0\}$. Hence, if we let D_g be the domain of g and R_f is the range of f, then $D_g \cap R_f = \emptyset$. Therefore, $g \circ f$ is undefined. ■

10. (Modified) Let f and g be functions defined by $f(x) = \sqrt{2-x}$, $x \leq 2$ and

   $$g(x) = \begin{cases} x^2 - 1, & x \in (-\infty, -1] \\ \frac{1}{x}, & x \in (-1, \infty). \end{cases}$$

   Find $g \circ f$ and its domain.

Solution: Notice that $\sqrt{2-x} \geq 0$ for $x \leq 2$, i.e. the range of f is $[0, \infty)$. From the formula of g, the domain of g is $\mathbb{R} - \{0\}$. Hence, if we let D_g be the domain of g and R_f is the range of f, then $D_g \cap R_f = (0, \infty) \neq \emptyset$. Therefore, $g \circ f$ can be defined as follows.

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{2-x}) = \frac{1}{\sqrt{2-x}}.$$

Note that we have used the fact that $f(x) = \sqrt{2-x} \geq 0$ so that the second formula of g was used. Notice that we must use $x < 2$ for $(g \circ f)(x)$. Hence the domain of $g \circ f$ is $(-\infty, 2)$. ■