Chapter 2 Matrices

2.1 Matrices

Definition 2.1 Let m and n be positive integers. A ***matrix*** of order $m \times n$ is an array of elements arranged in m rows and n columns. The elements of a matrix can be any mathematical objects: real numbers, complex numbers, functions, or itself a matrix. The elements are sometimes called ***members*** or ***entries*** of the matrix.

We will confine ourselves to matrices of real numbers or real-valued functions. A matrix will always be denoted by a bold upper-case letter. A matrix $\mathbf{A}$ of order $m \times n$ can be written explicitly as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

This matrix can be written as $\mathbf{A}_{m\times n}$ with subscript to specify its order. If is its generic element in row i and column j, we can also write,

$$\mathbf{A} = \left[a_{ij}\right]_{m\times n}.$$

We can also write $\mathbf{A} = \left[a_{ij}\right]$ with its order suppressed when there is no risk of confusion.

Definition 2.2 A matrix with only one column (row) and n rows (columns) is called a ***column (row) vector***. Unless specified otherwise, a vector means column vector and is denoted by a bold lower-case letter.

Example A vector $\mathbf{v}$ is given by $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ x \end{bmatrix}$.

A vector $\mathbf{a}$ with n elements can be interpreted as a point in Euclidean space $\boldsymbol{R}^n$, and write $\mathbf{a} \in \boldsymbol{R}^n$. Analogously, we can say that matrix $\mathbf{A}_{m\times n}$ belongs to $\boldsymbol{R}^{m\times n}$.

2.2 Matrices and System of Linear Equations

Given a system of linear equations with n variables and m equations, the ***coefficient matrix*** is the matrix $\mathbf{A}$ whose elements are the coefficients of the system of linear equations. That is,

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The RHS of the system is then just a column vector **b**, where

$$\mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}\in \boldsymbol{R}^m .$$

Definition 2.3 Given the coefficient matrix $\mathbf{A}\in \boldsymbol{R}^{m\times n}$ and the RHS $\mathbf{b}\in \boldsymbol{R}^m$, the ***augmented matrix*** is given by,

$$\hat{\mathbf{A}}=\begin{bmatrix}\mathbf{A} & \mathbf{b}\end{bmatrix}\in \boldsymbol{R}^{m\times(n+1)}.$$

That is, the first *n* columns of $\hat{\mathbf{A}}$ are just those of matrix **A** and the last column is **b**. Thus any system of linear equations can be completely represented by its augmented matrix.

We can define three types of elementary row operations on the rows of a given matrix in exactly the same way as in Definition 1.2 for the system of linear equations. The elementary row operations can then be applied to rows of $\hat{\mathbf{A}}$ to solve a system of linear equations. The elementary row operations create a series of augmented matrices that represent equivalent systems of linear equations. The final simplified matrix will be one that is in a form called row echelon.

2.2.1 Row Echelon Form and Reduced Row Echelon Form

Definition 2.4 A matrix $\mathbf{A}\in \boldsymbol{R}^{m\times n}$ is in ***row echelon form*** if,

a) suppose row *i* does not consist entirely of zeros, the number of leading zero element of row $(i+1)$ is greater than that of row *i*, for $i=1,2,\ldots,m-1$, and

b) suppose row *i* consists entirely of zeroes, so must rows $i+1,i+2,\ldots,m$.

A matrix that is in row echelon form is called a ***row echelon matrix***.

Example The matrices below are in row echelon form:

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \text{and} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

but the matrices $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & 0 \end{bmatrix}$ are not.

Example Find a row echelon form of the matrix $\begin{bmatrix} 2 & 6 & 8 \\ 8 & 2 & 2 \\ -4 & 1 & 5 \end{bmatrix}$.

Similar to what we define for the system of linear equations, we can call the first nonzero element, from the left, in each row of the row echelon matrix the ***pivot element***.

Definition 2.5 A matrix is in ***reduced row echelon form*** if it is a row echelon matrix with each pivot element being one and all the other elements in the column of that pivot element are zeroes.

Example The following are reduced row echelon matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{and} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem Given a row echelon matrix, describe how to convert it into reduced row echelon form.

Proposition 2.1 Given a row echelon matrix $\mathbf{A} \in \boldsymbol{R}^{m \times n}$, with its first k rows being nonzero and the last $m-k$ equations being zeros, $k = 0,1,\ldots,m$, the elementary row operations cannot reduce any of the first k rows into a row of all zeros.

Proof Exercise.❑

Theorem 2.1 All the row echelon forms of a given matrix $\mathbf{A} \in \boldsymbol{R}^{m \times n}$ always have the same number of nonzero rows.

Proof Suppose that there are two distinct series of elementary row operations that produce $\mathbf{A}, \mathbf{B}_1, \ldots \mathbf{B}_i, \mathbf{U}_1$ and $\mathbf{A}, \mathbf{C}_1, \ldots \mathbf{C}_j, \mathbf{U}_2$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are row echelon matrices. Suppose also that the number of nonzero rows of $\mathbf{U}_1$ is more than that of

$\mathbf{U}_2$. Thus, there is a series of elementary row operations that can be performed on the matrix $\mathbf{U}_1$ to produce another echelon matrix, namely, $\mathbf{U}_2$ with less number of nonzero rows. Why? This directly contradicts Proposition 2.1, and the theorem holds.❑

2.3 Rank of a Matrix

Although there can be many row echelon forms of a given matrix, but they all have one thing in common: the number of nonzero rows. This quantity is crucial in the analysis of matrix and system of linear equations, and it is defined formally as,

Definition 2.6 The ***rank of a matrix*** $\mathbf{A} \in \boldsymbol{R}^{m \times n}$, denoted by ***rank*** **A**, is the number of nonzero rows of the row echelon matrix resulted from performing elementary row operations on matrix **A**. The matrix **A** is ***full rank*** if rank **A** is *m*.

Proposition 2.2 For a matrix $\mathbf{A} \in \boldsymbol{R}^{m \times n}$, $0 \le \text{rank } \mathbf{A} \le \min\{m, n\}$.

Proof Since *m* is the number of rows, rank **A** by its definition cannot be greater than *m*. If $m \le n$, then the proposition holds. Suppose that $m > n$, rank **A** can be biggest if the resulting row echelon matrix has the pivot elements in row 1 and column 1, and row 2 and column 2, and so on. That is, the pivot elements form the perfect diagonal pattern. Since there are only *n* columns, the pivot elements can be in only *n* rows and the others are zero rows. Then $0 \le \text{rank } \mathbf{A} \le n < m$ and this proves the proposition.❑

Proposition 2.3 Let $\mathbf{A} \in \boldsymbol{R}^{m \times n}$ and $\mathbf{b} \in \boldsymbol{R}^{m}$. Then,

$$\text{rank } \mathbf{A} \le \text{rank} \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix} \le \text{rank } \mathbf{A} + 1.$$

Proof Since $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$ contains matrix **A** itself, when row operations are performed on $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$ to obtain a row echelon form, we also have a row echelon form for **A**, which is just the first *n* columns of the row echelon form of $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$. So we have the first part that $\text{rank } \mathbf{A} \le \text{rank} \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$.

The extra column **b** can create at most one more nonzero row of the row echelon form of $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$. Why? Thus we have the second inequality.❑

2.4 Rank and Solutions of System of Linear Equations

We can now state, using the matrix terminology we discussed so far, the same conclusion that we have in the analysis of system of linear equations in the previous chapter.

Theorem 2.2 Let $\mathbf{A} \in \boldsymbol{R}^{m \times n}$ be coefficient matrix and $\mathbf{b} \in \boldsymbol{R}^{m}$ be the RHS vector of a system of linear equations. Then

a) The system of linear equations has no solution if, and only if, $\operatorname{rank} \mathbf{A} < \operatorname{rank} \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$.

b) There is a unique solution if, and only if, $\operatorname{rank} \mathbf{A} = \operatorname{rank} \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix} = n$, and there are infinite number of solutions if, and only if, $\operatorname{rank} \mathbf{A} = \operatorname{rank} \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix} < n$.

Proof This theorem simply restates Theorem 1.1 and Corollary 1.1 in terms of rank.❑

Definition 2.7 A solution to a system of linear equations is ***trivial*** if the values of all variables are zeros.

Recall the definition of the homogeneous system of linear equations that is homogeneous if $\mathbf{b} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \boldsymbol{R}^{m}$.

Corollary 2.1 A homogeneous system of linear equations always has a solution, and if the solution is unique, it must be trivial.

Proof Any homogeneous system of linear equations always has a trivial solution. It then also follows that if it has a unique solution, that solution must be trivial.❑

Problem Show that if $\mathbf{x}$ and $\mathbf{y}$ are solutions to a homogeneous system of linear equations, so are $c\mathbf{x}$ and $\mathbf{x} + \mathbf{y}$, for any scalar *c*. See Definition 3.2 and 3.3 scalar multiplication and addition of matrices.

Problem Fraleigh & Beauregard [1995], page 69, #29 e-j.
29. Mark each of the folliwng True or False.

______**e)** Every matrix is row equivalent to a unique matrix in row-echelon form.

______**f)** Every matrix is row equivalent to a unique matrix in reduced row-echelon form.

______**g)** If $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{B} & \mathbf{c} \end{bmatrix}$ are row equivalent augmented matrices, the linear systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Bx} = \mathbf{c}$ have the same solution set.

______**h)** A linear system with a square coefficient matrix **A** has a unique solution if and only if **A** is row equivalent to the identity matrix.

______**i)** A linear system with coefficient matrix **A** has an infinite number of solutions if and only if **A** can be row-reduced to an echelon matrix that includes some column containing no pivot.

______**j)** A consistent linear system with coefficient matrix **A** has an infinite number of solutions if and only if **A** can be row-reduced to an echelon matrix that includes some column containing no pivot.

***Problem** Let **A** be a matrix of order $m \times n$. Suppose that **B** is a matrix of order $m \times (n+1)$ and given by $\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{a}_{n+1} \end{bmatrix}$, where $\mathbf{a}_{n+1} \neq \mathbf{0}$. That is, the matrix **B** has the first *n* columns being identical to the *n* columns of **A**. How would the two systems of linear equations $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{By} = \mathbf{b}$ be related? Answer each of the following cases.

a) If $\mathbf{Ax} = \mathbf{b}$ has a unique solution, can $\mathbf{By} = \mathbf{b}$ have a unique solution?

b) If $\mathbf{Ax} = \mathbf{b}$ has a unique solution, can $\mathbf{By} = \mathbf{b}$ be inconsistent?

c) If $\mathbf{Ax} = \mathbf{b}$ has a unique solution, can $\mathbf{By} = \mathbf{b}$ have an infinite number of solutions?

d) If $\mathbf{Ax} = \mathbf{b}$ is inconsistent, can $\mathbf{By} = \mathbf{b}$ have a solution? Unique? Infinite?

e) If $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solution, can $\mathbf{By} = \mathbf{b}$ have a unique solution?

f) If $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solution, can $\mathbf{By} = \mathbf{b}$ also have an infinite number of solutions?

g) If $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solution, can $\mathbf{By} = \mathbf{b}$ be inconsistent?

***Problem** Let **A** be a matrix of order $m \times n$. Suppose that **B** is a matrix of order $(m+1) \times n$ and given by $\mathbf{B} = \begin{bmatrix} \mathbf{A} \\ \mathbf{a}_{m+1} \end{bmatrix}$, where $\mathbf{a}_{m+1} \neq \mathbf{0}$. That is, the matrix **B** has the first *m* rows being identical to the *m* rows of **A**. How would the two systems of linear equations $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{By} = \mathbf{b}' = \begin{bmatrix} \mathbf{b} \\ b_{m+1} \end{bmatrix} \in \boldsymbol{R}^{m+1}$ be related? Answer each of the following cases.

a) If $\mathbf{Ax} = \mathbf{b}$ has a unique solution, can $\mathbf{By} = \mathbf{b}$ have a unique solution?

b) If $\mathbf{Ax} = \mathbf{b}$ has a unique solution, can $\mathbf{By} = \mathbf{b}$ be inconsistent?

c) If $\mathbf{Ax} = \mathbf{b}$ has a unique solution, can $\mathbf{By} = \mathbf{b}$ have an infinite number of solutions?

d) If $\mathbf{Ax} = \mathbf{b}$ is inconsistent, can $\mathbf{By} = \mathbf{b}$ have a solution? Unique? Infinite?

e) If $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solution, can $\mathbf{By} = \mathbf{b}$ have a unique solution?

f) If $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solution, can $\mathbf{By} = \mathbf{b}$ also have an infinite number of solutions?

g) If $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solution, can $\mathbf{By} = \mathbf{b}$ be inconsistent.