B.E. International Program
Faculty of Economics, Thammasat University

EE 320 Introductory Mathematical Economics

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Homework# 4

Solution

Question 1

Consider the function f defined for all (x,y) such that

$$f\ (x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2\ ,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1 - a^3, a^2)$ is a stationary point of $f\ (x, y; a)$.

FOC: $[x]$: $x - 1 + ay = 0$

$$[y]: a(x-1) - y^2 + 2a^2y = 0$$

To prove that $(x^*, y^*) = (1 - a^3, a^2)$ is "a stationary point, we must show that (x^*, y^*) satisfies the first-order conditions. Evidently, if you plug the given $(x^*, y^*) = (1 - a^3, a^2)$ into the two equations above, both conditions hold. Thus, $(1 - a^3, a^2)$ is a stationary point.

Don't you have to try figuring out yourself all the ordered pairs that solve for the two FOCs.

b. Given that $G(a) = f\ (x^*, y^*; a)$, show the derivative of G with respect to a.

$G(a) = f\ (x^*, y^*; a) \qquad = \frac{1}{2}(1-a^3)^2 - (1-a^3) + a * a^2(1-a^3 - 1)\ - \frac{1}{3}(a^2)^3 + a^2(a^2)^2$

$$= \frac{1}{2}(1-a^3)^2 - (1-a^3)\ - \frac{1}{3}(a^6)$$

$$\frac{dG}{da} = (1-a^3)(-3a^2) + 3a^2 - 2a^5 = a^5$$

c. Calculate $\frac{\partial f\ (x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1-a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?

$$\frac{\partial f\ (x,y;a)}{\partial a} = y(x-1) + 2ay^2$$

Plug $(x^*, y^*) = (1-a^3, a^2)$ into the above equation, we yield that:

$$a^2(1-a^3-1) + 2a(a^2)^2\ = a^5$$

The equivalent results as obtained from (b) and (c) are formally known under *the Envelop theorem*. Heuristic-based proof is given in this example. (See the rigorous proof in Chiang.) In general, the theorem states that the order in which you differentiate the function with respect to "a" doesn't matter. At the optimal solution, both methods *must* result in the same answer.

d. Where in the xy-plan is f convex.

Consider the second-order derivative matrix:

$$H = \begin{bmatrix} 1 & a \\ a & -2y + 2a^2 \end{bmatrix}$$

$|H1| = 1$, and thus is guaranteed to be greater than zero. We only need to establish a condition that ensures $|\mathrm{H2}|$ is positive. From the calculation, we know that

|H2| $= -2y + a^2 > 0.$ This term is evidently positive when $y < \frac{a^2}{2}$.

Question 2: Price discrimination (Moderate)

Suppose that there are three groups of people who take Sky train to commute in Bangkok. The first group is students (s), the second group is senior citizens (old), and the third group is working-aged people (w). The demand for each group is given by the following equations:

Demand of students:	$\boldsymbol{p_s = 8 - Q_s/2}$
Demand of senior citizens:	$\boldsymbol{p_{old} = 16 - 2Q_{old}}$
Demand of working-aged people:	$\boldsymbol{p_w = 20 - Q_w}$

The Sky train operator has a constant marginal cost at MC = \$4, and total cost at TC = 4Q + 10. Consider the following problems.

a) Determine the profit-maximizing level of output/price under the third-degree price discrimination. Calculate the level of maximized profit.

$$\pi = p_s Q_s + p_{old} Q_{old} + p_w Q_w - 4(Q_s + Q_{old} + Q_w) - 10$$

$$\pi_s = 8 - Q_s - 4 = 0 --> Q_s^* = 4 ---> p^*_s = 6$$
$$\pi_{old} = 16 - 4Q_{old} - 4 = 0 --> Q_{old}^* = 3 --> p^*_{old} = 10$$
$$\pi_w = 20 - 2Q_w - 4 = 0 ---> Q_w^* = 8 --> p_w^* = 12$$

Maximized profit is equal to 24 + 30 + 96 - 4(15) - 10 = 150 - 70 = 80

b) Confirm your result in (a) using the Hessian-matrix method.

$$\text{H} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

|H1| = -1 ; |H2| = 4; |H3| = -8

Question 3

Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:
Demand: $p_A = 10 - 2Q_A$
Supply: $p_A = 1 + Q_A$

Market B:
Demand: $p_B = 20 - Q_B$
Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium

Each of the two markets is to be sold for equilibrium separately. (Setting demand and supply equal to each other, for each of the two markets.)
This yields us $Q_A = 3$ and $P_A = 4$ for the market A. For market B, we yield that $Q_B = 6$ and $P_B = 14$

b. Suppose the government imposes unit tax on both markets at the rate of $t_A$and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.

For simplicity, we assume that tax is being imposed on consumers. (For the purpose of using the results obtained in "b" and working on the next problem "c", this assumption doesn't matter.)

So, ${p_A}^d = {p_A}^s + t_A$ and ${p_B}^d = {p_B}^s + t_B$. We yield that, after-tax equilibrium output would be $Q_A = \frac{9-t_A}{3}$ and $Q_B = \frac{18-t_B}{3}$. You can solve for the rest buy plugging these two Q back into the supply equation of each of the two markets. Doing so would give you producer's price, which then can be used to calculate for the consumers' price. To do so, just add producer's price with the unit tax imposed in each market.
$P_A^* = \frac{12-t_A}{3}$, $P_B^* = \frac{42-2t_B}{3}$

c. How much revenue can the government collect from the taxation?

$$TR = t_A Q_A + t_B Q_B = t_A\left(\frac{9 - t_A}{3}\right) + t_B\left(\frac{18 - t_B}{3}\right)$$

d. Determine the level of t_A and t_B that maximizes government's revenue.

Using the first order conditions approach, we yield that

$$[t_A]: 9 - (2)t_A = 0 \implies t_A = 4.5$$
$$[t_B]: 18 - (2)t_B = 0 \implies t_B = 9$$

<u>Question 4</u> (Monopolist and the Optimal advertising)

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. Both θ and β are strictly positive.

The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.

$$\pi = (2000 + 4\sqrt{A} - 20P)(P - 2) - A - 1000$$

b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.

$$\pi_A = 2(P-2)/\sqrt{A} - 1 = 0$$

$$\pi_P = 2040 + 4\sqrt{A} - 40P = 0$$

$P^* = 63.25$ and $A^* = 15{,}006.25$

c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.

$$H = \begin{bmatrix} -40 & \frac{2}{\sqrt{A}} \\ \frac{2}{\sqrt{A}} & \frac{-(P-2)}{A\sqrt{A}} \end{bmatrix}$$

|H1| = - 40 < 0;

|H2| = 40(P-2) / $A\sqrt{A}$) - (4 / A) > 0 ;
where $P^* = 63.25$ and $A^* = 15{,}006.25$

<u>Question 5</u> (Optimal input decision)

Suppose that the output Q of a firm depends on two inputs of the quantities K and L. The output level is determined by the production function

$$Q = 36K + 16L - 3K^2 - 2KL - L^2$$

a. Is the firm's production function strictly concave? Explain.

Ans.

$$Q_K = 36 - 6K - 2L$$

$$Q_L = 16 - 2K - 2L$$

$Q_{KK} = -6;\ Q_{LL} = -2;\ Q_{KL} = Q_{LK} = -2$

$|H_1| = -6 < 0$ and $|H| = Q_{KK}Q_{LL} - (Q_{KL})^2 = 12 - 4 = 8 > 0$

Hence, the production function Q(K,L) is a strictly concave function.

b. Determine the optimal input (K*, L*) that maximizes the output level.

Ans. K*=5, L*=3

c. Write down the firm's profit function when the price of *Q* is *P* and the per-unit factor prices of *K* and *L* are *r* and *w*, respectively, where both *r* and *w* are positive numbers. Find the levels of $K^*$and L^* that maximize the firm's profits.

Ans.

$$\pi(K,L) = P(36K + 16L - 3K^2 - 2KL - L^2) - rK - wL$$

$$Max_{K,L}\ \pi(K,L)$$

FONC:

$$\pi_K = P(36 - 6K - 2L) - r = 0 \qquad \text{--(1).}$$

$$\pi_L = P(16 - 2K - 2L) - w = 0 \qquad \text{--(2)}$$

From (1)&(2),

$$K^* = \frac{20P-r+w}{4P};\ L^* = \frac{12P-3w+r}{4P}$$

d. Verify that the second-order sufficient conditions for maximum profits are satisfied.

Ans.

SOSC:

$$\pi_{KK} = -6P < 0;\ \pi_{LL} = -2P < 0;\ \pi_{KL} = -2P.$$

$$\pi_{KK}\pi_{LL} - (\pi_{KL})^2 = 12P^2 - 4P^2 = 8P^2 > 0$$

Hence, the SOSC for a maximum profit is satisfied.

e. Determine the effect of an increase in *r* on the firm's use of each input. (i.e. determine $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial r}$).

Ans.

$\frac{\partial K^*}{\partial r} = -\frac{1}{4p} < 0; \frac{\partial L^*}{\partial r} = \frac{1}{4p} > 0$

<u>Question 6</u>

Each of two skin clinics, *W* and *N*, provides its own brand of a skin treatment in the amounts Q_w and Q_n at the prices P_w and P_n, respectively. Each clinic independently chooses the quantity of treatment that maximizes its own profit, where the demand functions for the two brands are given by:

$$P_w = 60 - 6Q_w + 2Q_n$$

$$P_n = 30 + 2Q_w - 4Q_n$$

Suppose that the total cost functions for clinics W and N are:

$$C_w = 18 + 8Q_w$$

$$C_n = 24 + 8Q_n$$

a. Initially each clinic maximizes its own profit, taking another clinic's optimal quantity as given. Determine the amounts Q^*_w and Q^*_n that maximize each clinic's profit, and calculate the corresponding profit level of each clinic.

W: $Max_{Q_w}\ \pi^w = (60 - 6Q_w + 2Q_n)Q_w - (18 + 8Q_w)$

FONC: $52 - 12Q_w + 2Q_n = 0$ -- (1)

N: $Max_{Q_n}\ \pi^n = (30 + 2Q_w - 4Q_n)Q_n - (24 + 8Q_n)$

FONC: $22 + 2Q_w - 8Q_n = 0$ -- (2)

From (1) & (2), $Q^*_w = 5$ and $Q^*_n = 4$

$\pi^*_w = 132$ and $\pi^*_n = 40$.

b. Suppose now that the two clinics collude and act as a monopolist whose total revenue is the combined revenues and total cost is the combined costs.

Determine the new amounts $Q^*{}_w$ and $Q^*{}_n$ that maximize the combined profit, and calculate the new maximum profit. Discuss the change in the overall profits, if any, after the two clinics collude.

Combined profit:

$$\pi^T = (60 - 6Q_w + 2Q_n)Q_w + (30 + 2Q_w - 4Q_n)Q_n - (18 + 8Q_w) - (24 + 8Q_n)$$

$$Max_{Q_w,Q_n}\ \pi^T$$

FONC: $\frac{\partial \pi^T}{\partial Q_w} = 52 - 12Q_w + 4Q_n = 0$

$$\frac{\partial \pi^T}{\partial Q_n} = 22 + 4Q_w - 8Q_n = 0$$

$\Rightarrow Q^*{}_w = 6.3$ and $Q^*{}_n = 5.9$

$\Rightarrow \pi^*{}_T = 186.7 > \pi^*{}_w + \pi^*{}_n = 172$

Hence, total profit increases after merging.