EE 411 (1/2018)
Microeconomic Anlaysis
Lecture 0
Reviewing Static Optimization

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Lecture 0: Optimization Review

- Topics: Appendix A2.3 and A2.4
- Constrained Optimization
- Optimum Value Function
- Please review JR Appendix A1-A2.1 (Set, functions, basic optimization)

Constrained Optimization

- 1. Three types of constraints:
  - equality, $g(x_1 , x_2) = x_1 + x_2 = 0$
  - non-negativity : $x_1 \geq 0$, $x_2 \geq 0$
  - inequality : $g(x_1 , x_2) = x_1 + x_2 \leq 0$.
- 2. Mostly we will focus on equality constraint problem.
- 3. Consider choosing x_1 and x_2 to maximize

  $f(x_1 , x_2)$ subject to $g(x_1 , x_2) = 0$.

  where x_1 and x_2 are called choice variables.
- 4. f is called the objective function.

Constrained Optimization

- 5. g is called the constraint set, or the feasible set.
- 6. The easiest way to solve this problem is by substitution if we can write x_2 in terms of x_1 from the g function.

$$\text{Max } f(x_1, h(x_1)); \; x_2 = h(x_1)$$

- FONC: $f_1 + f_2 h_1 = 0$ (using subscript for partial derivatives)
- Solve for x^*_1 and $x^*_2 = h(x^*_1)$.

Method of Lagrange for Constraint Optimization.

EX. Two variables with equality constraint.

$$\max \quad U(x_1, x_2)$$
$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 = I$$

Using geometry techniques, we know that

At (x_1^*, x_2^*), MRS = price ratio

Or $U_1 / U_2 = p_1 / p_2$

Slope of indifference curve=slope of budget line.

Slope of IC: using a given IC, say

$U(x_1, x_2) = 10 \Rightarrow U_1\, dx_1 + U_2 dx_2 = 0$ or

$$dx_2 / dx_1 = -\, U_1 / U_2$$

Slope of budget line: $p_2 x_2 = I - p_1 x_1$ or

$$x_2 = (I / p_2) - (p_1 / p_2)\, x_1$$

Method of Lagrange for Constraint Optimization

From $U_1 / U_2 = p_1 / p_2$

Let assume λ a positive value as

$$U_1 / p_1 = U_2 / p_2 = \lambda$$

and rearrange terms above along with constraint

$$U_1 - \lambda p_1 = 0$$

$$U_2 - \lambda p_2 = 0$$

$$p_1 x_1 + p_2 x_2 = I$$

With 3 equations, we can solve for 3 unknowns (x_1 , x_2 , λ).

Writing Lagrange Equation

Let the new function L a function of 3 variables:

$L(x_1 , x_2 , \lambda) \equiv U (x_1 , x_2) + \lambda [I - p_1 x_1 - p_2 x_2]$

FONC: (partial derivatives of L wrt each variable =0)

$L_1 = 0 ; \quad => \quad U_1 - \lambda p_1 = 0$

$L_2 = 0 ; \quad => \quad U_2 - \lambda p_2 = 0$

$L_\lambda = 0 ; \quad => \quad p_1 x_1 + p_2 x_2 = I$

These necessary conditions are exactly similar to the previous conditions.

We can deal with more constraints by introducing more Lagrange multipliers.

What is the Lagrange Multiplier?

- λ is called "Lagrange Multiplier" or the shadow price (imputed value) of the constraint.
- In economics, it is the price of resource constraint. So, we will make $[I - p_1 x_1 - p_2 x_2]$ in L function to be positive, and λ will always have a positive value.
- Ex. Max $U(x_1, x_2) = x_1 x_2$ st. $x_1 + 4x_2 = 16$

  answer: $x_1^* = 8$, $x_2^* = 2$, $\lambda = 2$.

  note: $MRS = U_1 / U_2 = x_2 / x_1$,

  Price ratio=1/4.

Checking for Sufficient Conditions (SOC)

- Consider first the optimization problem without constraint.
- FONC: "If x* maximizes f, then f'(x*)=0". (reverse not true)
- We know x* could be a minimum too.
- In other words, this is only a necessary condition for maximum.
- We need a sufficient condition (SOC) to make sure that we get the right answer. "If such and such.. .obtains at x*, then x* optimizes the function."
- SOC: if f'(x*)=0 and H(x*) is negative definite at x*, then f(x) reaches a local maximum at x*."
- Strict forms of the curve rules out inflection point for an optimum.

Checking for Sufficient Conditions (SOC)

- Intuitively, when we check the second-order partial derivatives or H(X), we look at the curvature of the objective function.
- When the f function is strictly concave at x*, we know that H(x*) is negative definite. (or f″ < 0 for a function of one variable)
- Example for a strictly concave function

(a) f(x)= ln(x). Check $f'' = -x^{-2} < 0$.

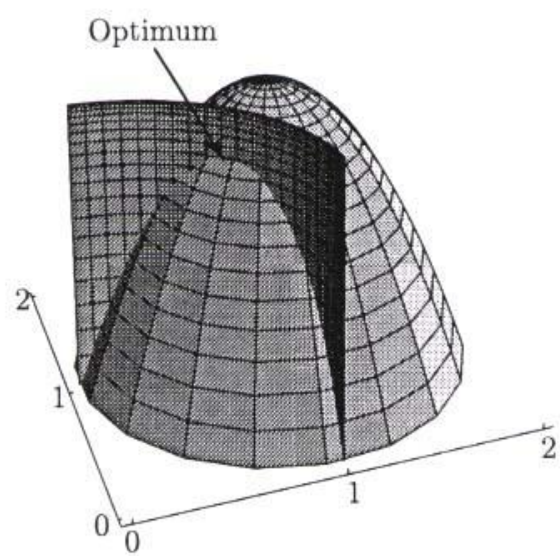
(b) $f(x) = x_1^{0.5} x_2^{0.5}$

$$\mathbf{H(x)} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix},$$

$$need\ f_{11} < 0\ and\ |H| > 0$$

Checking for Sufficient Conditions (SOC)

- Now, consider constrained optimization.
- SOC requires us to check the bordered H(x).
- We examine the curvature of the objective function at the optimum along the constraint.



The optimum is the highest point that is common to the objective surface and the constraint

Checking for Sufficient Conditions (SOC)

- Sufficient condition for constrained maximum of two variables and one constraint requires determinant of the bordered Hessian to be > 0.
- The bordered H(x) of problem with 2 variables and one constraint can be defined as
- $$\bar{\mathbf{H}}(\mathbf{x}) = \begin{bmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{bmatrix}$$

$$L(\lambda, x_1, x_2) \equiv f(x_1, x_2) - \lambda g(x_1, x_2)$$

$$\frac{\partial L}{\partial x_1} = f_1 - \lambda g_1 = 0; \Rightarrow \frac{\partial^2 L}{\partial x_1^2} = f_{11} - \lambda g_{11}.$$

$$\frac{\partial L}{\partial x_2} = f_2 - \lambda g_2 = 0; \Rightarrow \frac{\partial^2 L}{\partial x_2^2} = f_{22} - \lambda g_{22}.$$

$$\frac{\partial L}{\partial \lambda} = g(\mathbf{x}) = 0; \qquad \Rightarrow \frac{\partial^2 L}{\partial \lambda^2} = 0.$$

$$\frac{\partial^2 L}{\partial \lambda \partial x_1} = -g_1; \frac{\partial^2 L}{\partial \lambda \partial x_2} = -g_2$$

$$\bar{H} = \begin{pmatrix} \frac{\partial^2 L}{\partial \lambda^2} & \frac{\partial^2 L}{\partial \lambda \partial x_1} & \frac{\partial^2 L}{\partial \lambda \partial x_2} \\ \frac{\partial^2 L}{\partial x_1 \partial \lambda_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial \lambda_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 0 & -g_1 & -g_2 \\ -g_1 & L_{11} & L_{12} \\ -g_2 & L_{21} & L_{22} \end{pmatrix}$$

$$L(\lambda, x_1, x_2) \equiv \ln(x_1) + \ln(x_2) + \lambda[I - p_1 x_1 - p_2 x_2]$$

$$\frac{\partial L}{\partial x_1} = 1 / x_1 - \lambda p_1 = 0 ; \Rightarrow \frac{\partial^2 L}{\partial x_1^2} = -1 / x_1^2 .$$

$$\frac{\partial L}{\partial x_2} = 1 / x_2 - \lambda p_2 = 0 ; \Rightarrow \frac{\partial^2 L}{\partial x_2^2} = -1 / x_2^2 .$$

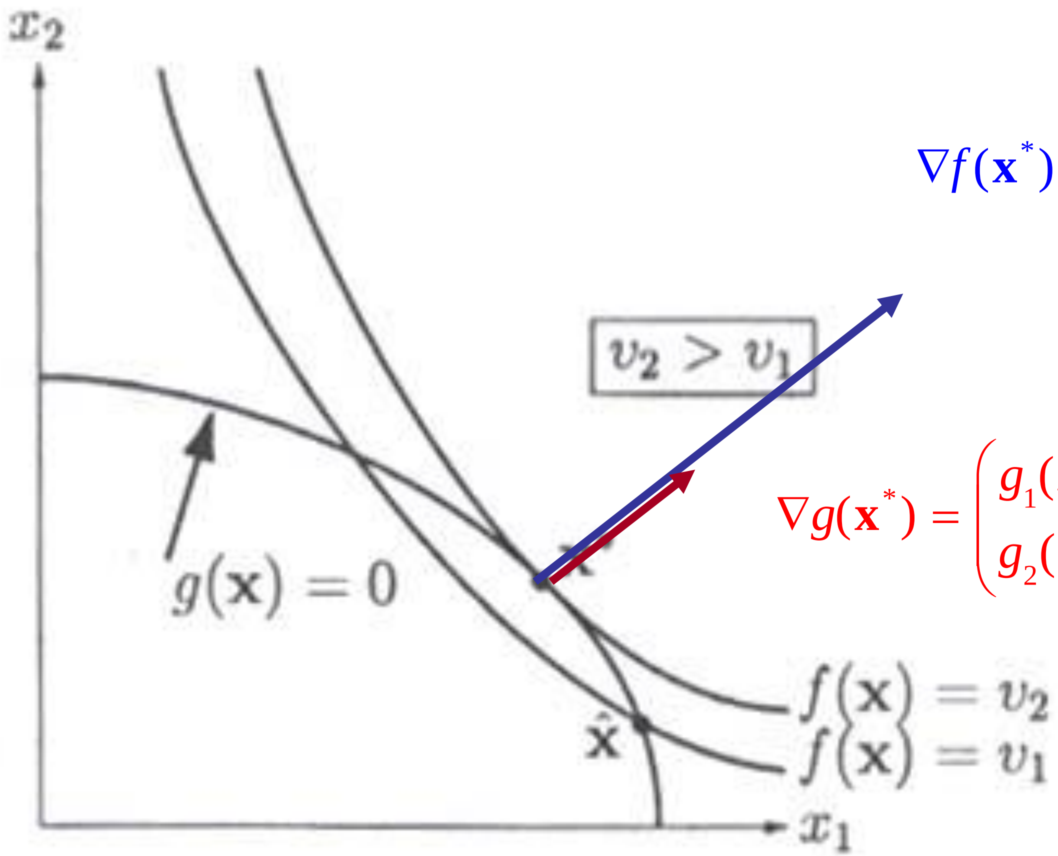
$$\frac{\partial L}{\partial \lambda} = I - p_1 x_1 - p_2 x_2 = 0; \Rightarrow \frac{\partial^2 L}{\partial \lambda^2} = 0.$$

$$\frac{\partial^2 L}{\partial \lambda \partial x_1} = -p_1 ; \frac{\partial^2 L}{\partial \lambda \partial x_2} = -p_2$$

$$\bar{H} = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & \dfrac{-1}{x_1^2} & 0 \\ -p_2 & 0 & \dfrac{-1}{x_2^2} \end{pmatrix},$$

$$|\bar{H}| = p_1(-p_1)(-1 / x_2^2) - p_2(-1)(-p_2)(-1 / x_1^2)$$
$$= (p_1 / x_2)^2 + (p_2 / x_1)^2 > 0.$$

$$\nabla f(\mathbf{x}^*) = \begin{pmatrix} f_1(\mathbf{x}^*) \\ f_2(\mathbf{x}^*) \end{pmatrix}$$



$$\nabla g(\mathbf{x}^*) = \begin{pmatrix} g_1(\mathbf{x}^*) \\ g_2(\mathbf{x}^*) \end{pmatrix}$$

Tangency between the constraint and the objective function

Summary of Lagrangean Method

- 1. Forming the L function and make sure that the term after lambda is positive. You can write it many ways:

$$L(x_1, x_2, \lambda) \equiv f(x_1, x_2) - \lambda\, g(x_1, x_2)$$

or $$= f(x_1, x_2) - \lambda\,(g(x_1, x_2) - c)$$

or $$= f(x_1, x_2) + \lambda\,[c - g(x_1, x_2)]$$

- 2. Set the partial derivative of the L function with respect to x_1, x_2, and λ equal to zero. Here, we get three equations.
- 3. To solve three equations for three unknowns, first get rid of λ (using the first and second equations), and write x_1 in terms of x_2, and then using this fact in the constraints or the third equation.

Summary of Lagrangean Method

- 4. Your optimal value of x_1, x_2 and λ must depend only on parameters.
- 5. When plugging these optimal values in the objective functions, we get the maximized value of the function.
- 6. The optimal value function is a function of parameters. We can find the maximized value by knowing these parameters without solving the problem.

Summary of Lagrangean Method

- 7. Second order conditions is to check the bordered Hessian.
- 8. SOC for a maximization problem with two variables and one constraint is to verify that the determinant of bordered Hessian > 0 evaluated at optimum.
- As for Min problem, SOC needs the determinant of the bordered Hessian < 0.
- Notice that if we multiply the first row and first column by -1, the sign of determinant of bordered H will not change.

Summary of Lagrangean Method

- 9. SOC for a maximization problem with n variables and k constraints is to verify that the last n-k of the leading principal minors of the bordered Hessian alternate in sign at optimum.
- As for Min problem, SOC needs the last n-k of the leading principal minors of the bordered Hessian are all negative.

Summary of Lagrangean Method

- How to remember this. Remind that
- A positive definite matrix is the identity Matrix. See that the leading principal minors of identity matrix are all positive.
- A negative definite matrix is the negative of the identity matrix. Now check that the leading principal minors of this matrix must alternate in sign.

$$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} > 0; \quad \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1 \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} < 0.$$

Summary of Lagrangean Method

- 9. The meaning of the Lagrange multiplier is known by using the envelope theorem. In utility maximization problem, it is a marginal utility of income. In cost minimization problem, it is a marginal cost.
- 10. When the Lagrange multiplier has a value (not equal zero), then this implies a shadow price of resources. Equivalently, it is to say that the constraint is binding, or resource is scarce and must be used up.

Value function

- Consider Max f(**x**, **a**) s.t. g(**x**, **a**) = 0
- We call "**a**" parameters
- Solution we get must be expressed in terms of **a**. That is, we have **x***=**x**(**a**)
- If we put **x(a)** in our objective function, we get M(**a**) ≡ f(**x**(**a**), **a**)**.**
- We call M(**a**) as **a maximum value function.**

EX. Max $U(x_1 , x_2) = x_1 x_2$

s.t. $x_1 + 4x_2 = a$

$L(x_1 , x_2 , \lambda) \equiv x_1 x_2 + \lambda [a - x_1 - 4x_2]$

FONC:

$L_1 = 0$; => $x_2 - \lambda = 0$ ....(1)

$L_2 = 0$; => $x_1 - \lambda 4 = 0$ ......(2)

$L_\lambda = 0$; => $a - x_1 - 4 x_2 = 0$ .....(3)

Using (1) and (2), we get $x_1 = 4 x_2$

Sub $x_1 = 4x_2$ in (3), we get $a - 4x_2 - 4 x_2 = 0$

Thus $x_2 * = a/ 8$; $x_1 * = a/ 2$; $\lambda * = a/ 8$

And $U(a) = x_1 * x_2 * = a^2/ 16$

Value function

- The value function gives the value when choice variables are chosen to maximize f subject to the constraints.
- M(**a**) tells us how the maximum value of function will change as the value of **a** changes.
- If we already knew M(**a**), we can do the calculation. From previous example,

  $$\partial U(a) / \partial a = a/8$$

- Note that it happens to be equal to λ^*

Value function

- If we want to know how the solutions to the maximization problem vary with parameters.
- We can redo the problem with a new parameter, or
- We can apply the envelope theorem.
- The Envelope theorem gives us a formula for the derivative of the value function w.r.t a parameter in the optimization problem.

Value function

- Formula for the Envelope theorem:

  Total effect on the optimized value of the objective function when a parameter changes can be calculated by

$$\frac{dM(a)}{da_i} = \left.\frac{\partial L}{\partial a_i}\right|_{x(a)}$$

- measure changes around the optimal values.

$$M(a) \equiv f(x_1(a), x_2(a), a)$$

$$\frac{dM(a)}{da_i} = \left.\frac{\partial L(x,a)}{\partial a_i}\right|_{x(a)} = \left.\frac{\partial f(x,a)}{\partial a_i}\right|_{x(a)} - \lambda \left.\frac{\partial g(x,a)}{\partial a_i}\right|_{x(a)}$$

$$Proof.\ assuming\ differentiability$$

$$\frac{dM(a)}{da_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial a_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial a_i} + \frac{\partial f}{\partial a_i}.$$

$$= \lambda \underbrace{\left[\frac{\partial g}{\partial x_1}\frac{\partial x_1}{\partial a_i} + \frac{\partial g}{\partial x_2}\frac{\partial x_2}{\partial a_i}\right]} + \frac{\partial f}{\partial a_i} \qquad use\ FOC$$

$$= \lambda \left[-\frac{\partial g}{\partial a_i}\right] + \frac{\partial f}{\partial a_i} \qquad use\ diff\ g(x_1(a), x_2(a), a) = o$$

Value function

- Idea: the value function changes because of two effects.
- First, the indirect effect, x(a) changes so that f(a) changes.
- Secondly, the direct effect of change in a.
- Since x(a) is optimal, so the indirect effect will have a negligible influence on f.

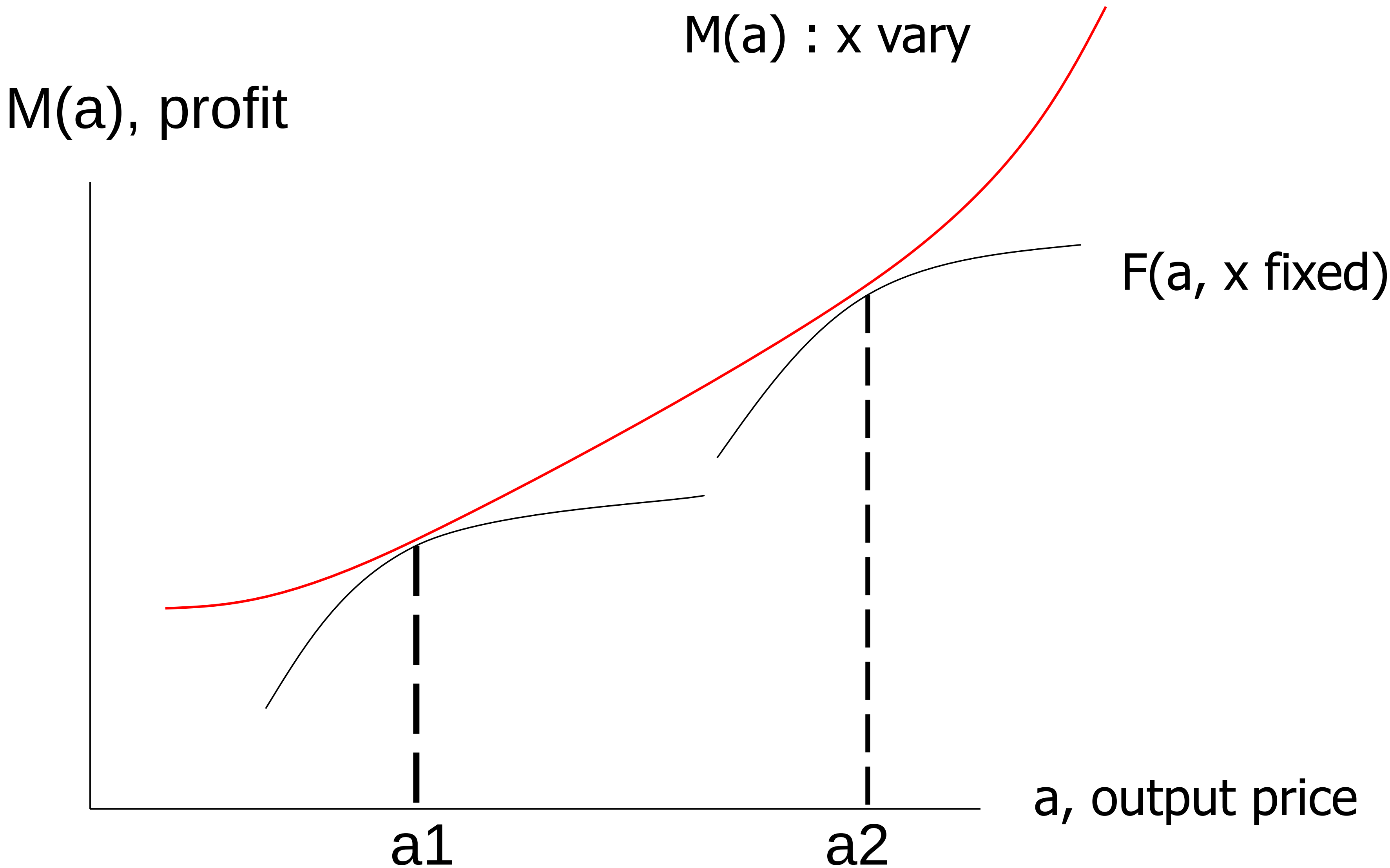
Value function

- From the previous example,
  $\partial U(a) / \partial a = a/8$.
- Envelop theorem,
  $\partial U(a) / \partial a = \partial L / \partial a$ evaluated at the solution of the original problem .
- From
  $\mathbf{L(x_1 , x_2 , \lambda) \equiv x_1 \, x_2 + \lambda \, [\, a - x_1 - 4x_2]}$
- $\partial L / \partial a = \lambda^* = a/8$
- Thus, we save times to construct the value function and do the partial derivative.

Value function : Examples

- In consumer problems: the indirect utility function, and expenditure function
- In producer problems: the profit function, and cost function.

Why called the Envelope



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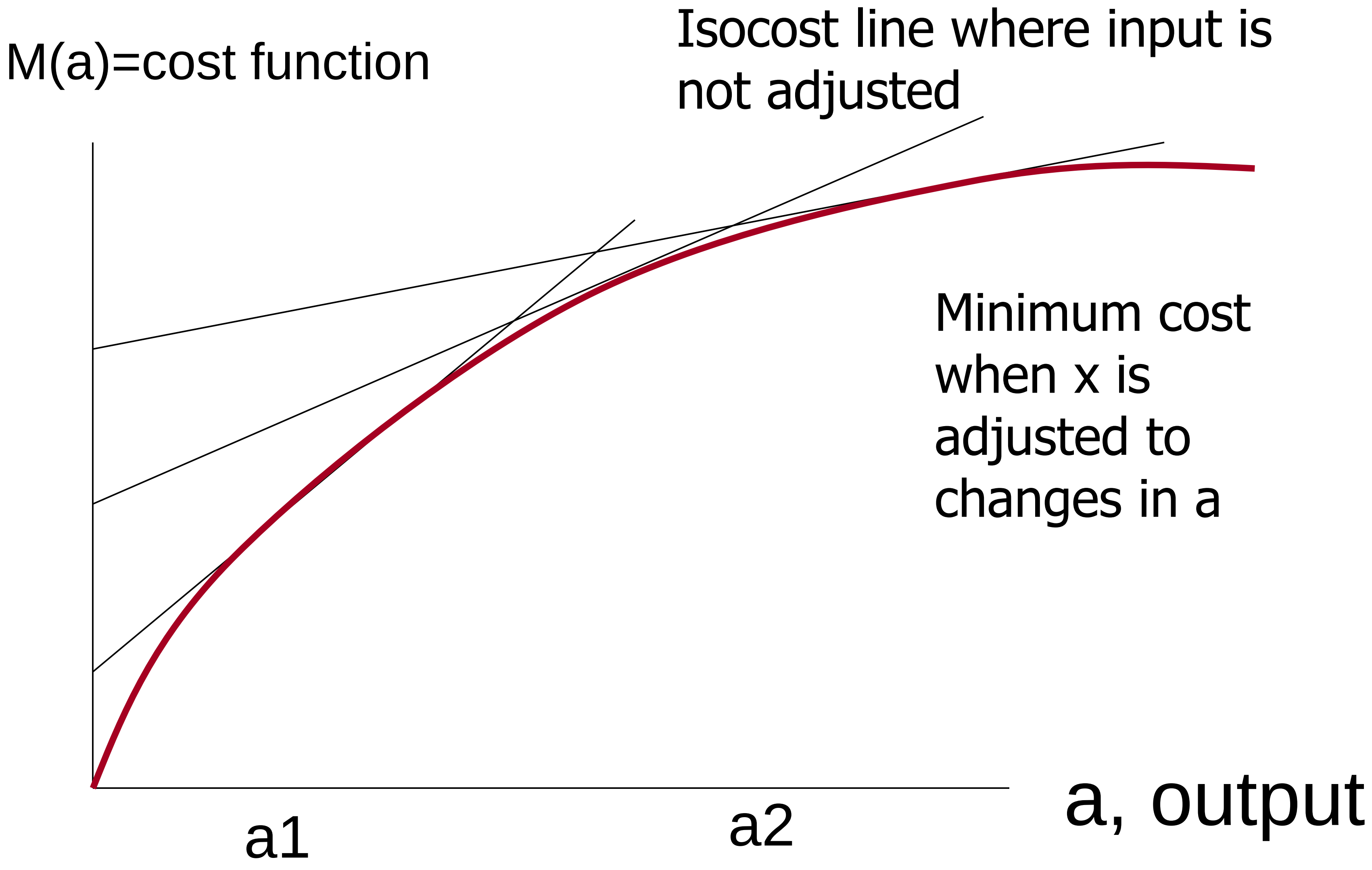

Optimizations in Economics

- 1. Indirect Utility function:

  $\max U(x_1, x_2)$ st. $p_1x_1 + p_2x_2 = I$.
- 2. Expenditure function:

  $\min \mathbf{px}$ st. $u(\mathbf{x})=c$.
- 3. Profit function:

  $\max py-\mathbf{wx}$ st. $f(\mathbf{x})=y$.
- 4. Cost function:

  $\min w_1x_1 + w_2x_2$ st. $f(x_1, x_2)=c$