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(Additional Notes)

Second-Order Conditions for Constrained Optimization Problem

$$\underset{x,y}{\text{Max (Min)}}\ f(x,y) \quad \text{subject to} \quad g(x,y) = c$$

① From constraint function : $g(x,y) = c.$

$$\Rightarrow dg = g_x dx + g_y dy = 0 \Rightarrow \boxed{dy = \left(-\frac{g_x}{g_y}\right)\cdot dx}$$

i.e. dy is a function of dx ; $dy = h(dx)$.

② From objective function : $z = f(x,y)$

$$dz = f_x dx + f_y dy \quad \text{where } dy = h(dx) = \left(-\frac{g_x}{g_y}\right) dx.$$

$$\Rightarrow d^2z = d(dz) = \frac{\partial}{\partial x}(dz)\cdot dx + \frac{\partial}{\partial y}(dz)\cdot dy$$

$$= \frac{\partial}{\partial x}(f_x dx + \underbrace{f_y}_{u}\,\underbrace{dy}_{v})\cdot dx + \frac{\partial}{\partial y}(f_x dx + \underbrace{f_y}_{u}\,\underbrace{dy}_{v})\cdot dy.$$

Need to use product rule because dy is now a variable dependent on x and y

$$= \left[f_{xx} dx + \left(\underbrace{f_y}_{u}\cdot \underbrace{\frac{\partial(dy)}{\partial x}}_{v'} + \underbrace{dy}_{v}\cdot \underbrace{f_{xy}}_{u'}\right)\right] dx$$

$$+ \left[f_{yx} dx + \left(f_y\cdot \frac{\partial(dy)}{\partial y} + dy\cdot f_{yy}\right)\right]\cdot dy$$

$$= f_{xx}\cdot dx^2 + \boxed{f_y\cdot \frac{\partial(dy)}{\partial x}\cdot dx} + f_{xy}\cdot dy\,dx$$

$$+ f_{yx}\,dx\,dy + \boxed{f_y\cdot \frac{\partial(dy)}{\partial y}\cdot dy} + f_{yy}\cdot dy^2$$

$$= f_{xx}\,dx^2 + 2f_{xy}\cdot dx\,dy + f_{yy}\cdot dy^2$$

$$+ \underbrace{\boxed{f_y\left[\frac{\partial(dy)}{\partial x}\cdot dx + \frac{\partial(dy)}{\partial y}\cdot dy\right]}}_{= f_y[d^2y]}$$

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$$\therefore \quad \boxed{d^2z = f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2 + f_y\cdot d^2y.} \quad \text{—} (*)$$

Note: Since $d^2y = d(dy) = \frac{\partial(dy)}{\partial x}.dx + \frac{\partial(dy)}{\partial y}.dy$,

$$f_y\left[\frac{\partial(dy)}{\partial x}.dx + \frac{\partial(dy)}{\partial y}.dy\right] = f_y\cdot d(dy) = f_y\cdot d^2y.$$

Recall: $dg = g_x\,dx + g_y\,dy = 0$

$$\Rightarrow d^2g = d(dg) = \frac{\partial(dg)}{\partial x}.dx + \frac{\partial(dg)}{\partial y}.dy = 0$$

$$(d^2g = 0 \Rightarrow) \quad \frac{\partial}{\partial x}(g_x\,dx + (g_y\,dy))\,dx + \frac{\partial}{\partial y}(g_x\,dx + (g_y\,dy))\,dy$$

$$0 = \left[g_{xx}\,dx + \left(g_y\cdot\frac{\partial(dy)}{\partial x} + dy\cdot g_{xy}\right)\right]dx$$

$$+ \left[g_{yx}\,dx + \left(g_y\cdot\frac{\partial(dy)}{\partial y} + dy.g_{yy}\right)\right]dy$$

$$0 = g_{xx}\,dx^2 + 2g_{xy}\,dx\,dy + g_{yy}\,dy^2 + g_y\underbrace{\left[\frac{\partial(dy)}{\partial x}.dx + \frac{\partial(dy)}{\partial y}.dy\right]}_{= d^2y}$$

$$\therefore d^2g = 0 = g_{xx}\,dx^2 + 2g_{xy}\,dx\,dy + g_{yy}\,dy^2 + g_y\cdot d^2y$$

$$\Rightarrow d^2y = -\frac{g_{xx}}{g_y}.dx^2 - \frac{2g_{xy}}{g_y}.dx\,dy - \frac{g_{yy}}{g_y}.dy^2$$

Substitute d^2y in $(*)$:

$$d^2z = f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2 + f_y\overbrace{\left[-\frac{g_{xx}}{g_y}dx^2 - 2\frac{g_{xy}}{g_y}.dx\,dy - \frac{g_{yy}}{g_y}.dy^2\right]}^{d^2y}$$

$$(**)$$

$$\therefore \boxed{d^2z = \left(f_{xx} - \frac{f_y}{g_y}.g_{xx}\right)dx^2 + 2\left(f_{xy} - \frac{f_y}{g_y}.g_{xy}\right)dx\,dy + \left(f_{yy} - \frac{f_y}{g_y}\,g_{yy}\right)dy^2}$$

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Recall FONC: (from $\mathcal{L} = f(x,y) + \lambda[c - g(x,y)]$)

FONC:
$$\left.\begin{aligned} \mathcal{L}_x &= f_x - \lambda g_x = 0 \\ \mathcal{L}_y &= f_y - \lambda g_y = 0. \end{aligned}\right\} \boxed{\lambda = \frac{f_x}{g_x} = \frac{f_y}{g_y}}$$
$$\mathcal{L}_\lambda = c - g(x,y) = 0$$

Second-order partial derivatives. :

$$\mathcal{L}_{xx} = f_{xx} - \lambda g_{xx} \Rightarrow \mathcal{L}_{xx} = f_{xx} - \overbrace{\frac{f_y}{g_y}}^{=\lambda} \cdot g_{xx}$$

$$\mathcal{L}_{yy} = f_{yy} - \lambda g_{yy} \Rightarrow \mathcal{L}_{yy} = f_{yy} - \overbrace{\frac{f_y}{g_y}}^{=\lambda} \cdot g_{yy}$$

$$\mathcal{L}_{xy} = \mathcal{L}_{yx} = f_{xy} - \lambda g_{xy} \Rightarrow \mathcal{L}_{xy} = f_{xy} - \overbrace{\frac{f_y}{g_y}}^{=\lambda} \cdot g_{xy}$$

Thus, equation (**) can be written as:

$$\boxed{d^2 z = \mathcal{L}_{xx} \cdot dx^2 + 2\mathcal{L}_{xy}\, dx\, dy + \mathcal{L}_{yy}\, dy^2}$$

In matrix form:

$$d^2 z = [dx \quad dy] \begin{bmatrix} \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

Second-order Conditions:

For a constrained extremum of $z = f(x,y)$ subject to $g(x,y) = c$, the second-order sufficient conditions are:

For maximum of z: $d^2 z$ is negative definite, subject to $dg = 0$

For minimum of z: $d^2 z$ is positive definite, subject to $dg = 0$.

The Bordered Hessian

To determine the sign definiteness of d^2z subject to a linear constraint, rewrite the sosc in a two-variable quadratic form as:

$$q = au^2 + 2huv + bv^2 \quad \text{subject} \quad \alpha u + \beta v = 0$$

where

$q \equiv d^2z$, $a \equiv \mathcal{L}_{xx}$, $b \equiv \mathcal{L}_{yy}$, $h \equiv \mathcal{L}_{xy}$, $u \equiv dx$, $v \equiv dy$,
$\alpha \equiv g_x$, and $\beta \equiv g_y$.

From the constraint $\alpha u + \beta v = 0$, $\quad v = -\left(\frac{\alpha}{\beta}\right)u$.

$$\Rightarrow q = au^2 + 2hu\left(-\frac{\alpha}{\beta}u\right) + b\left(-\frac{\alpha}{\beta}u\right)^2$$

$$\therefore \quad q = au^2 - 2h\frac{\alpha}{\beta}u^2 + b\frac{\alpha^2}{\beta^2}u^2 = (a\beta^2 - 2h\alpha\beta + b\alpha^2)\frac{u^2}{\beta^2}$$

$\Rightarrow$ q is positive definite iff $a\beta^2 - 2h\alpha\beta + b\alpha^2 > 0$;
q is negative definite iff $a\beta^2 - 2h\alpha\beta + b\alpha^2 < 0$.

Note that
$$\begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} = 2h\alpha\beta - a\beta^2 - b\alpha^2 = -(a\beta^2 - 2h\alpha\beta + b\alpha^2)$$

$$\boxed{\therefore\ q \text{ is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix} \text{ subject } \alpha u + \beta v = 0 \text{ iff } \begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} \begin{cases} < 0 \\ > 0 \end{cases}}$$

$$\begin{bmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{bmatrix} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$$ is called "Bordered Hessian" matrix. $|\bar{H}|$

Hessian: $H = \begin{bmatrix} \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$.

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Thus, for values of dx and dy that satisfy the constraint $g_x dx + g_y dy = 0$, the determinantal criterion for the sign definiteness of d^2z is:

- d^2z is positive definiteness s.t. $dg = 0$ iff $|\bar{H}| < 0$
- d^2z is negative definiteness s.t. $dg = 0$ iff $|\bar{H}| > 0$

where

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} .$$

$\therefore$ The SOSC for a constrained optimization problem Max (Min) $z = f(x,y)$ subject to $g(x,y) = c$ are:

$\Rightarrow$ For maximum of z : $d^2z < 0$ s.t. $dg = 0 \Leftrightarrow |\bar{H}| > 0$

For minimum of z : $d^2z > 0$ s.t. $dg = 0 \Leftrightarrow |\bar{H}| < 0$

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n - Variable Case

$\left.\begin{matrix}\text{Max} \\ \text{Min}\end{matrix}\right\}$ $z = f(x_1, x_2, \dots, x_n)$ subject to $g(x_1, \dots, x_n) = c$.

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \dots & g_n \\ g_1 & L_{11} & L_{12} & \dots & L_{1n} \\ \vdots & \vdots & \vdots & & \\ g_n & L_{n1} & L_{n2} & \dots & L_{nn} \end{vmatrix} \Rightarrow |\bar{H}_2| \equiv \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} .$$

$$|\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}, \dots$$

SOSC's are:

1) For maximum of z : $|\bar{H}_2| > 0$; $|\bar{H}_3| < 0$; $|\bar{H}_4| > 0$; ... ; $(-1)^n |\bar{H}_n| > 0$

2) For minimum of z : $|\bar{H}_2|, |\bar{H}_3|, \dots, |\bar{H}_n| < 0$.

Double A