1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02} f_2]$? Explain your answer.

A pricing using stochastic discount implies $S_0 = E_0(m_{01}, D_1) + E_0(m_{02}, S_2)$

$= D_0 + E_0(m_{02}, S_2)$

forward price $= F_{02} = f_2 = S_2 - F_{02}$ represent ownership in share of underlying asset, a short position the underlying asset's dividend and borrowing an amount such that the repayment at date 2 $= F_{02}$: We know that

$E_0(m_{02}, F_2) = E_0(m_{02}(S_2 - F_{02}))$

$= E(m_{02}, S_2) - E(m_{02} F_{02})$.

Note: $S_0 = E_0(m_{01} D_1) + E_0(m_{02} S_2) = D_0 + E_0(m_{02}, S_2)$ and $E(m_{02}) F_{02} = R_f^{-2} F_{02}$

We have $E_0(m_{02} F_2) = E(m_{02} S_2) - E(m_{02} S_2) - E(m_{02} f_{02})$

$= S_0 - D_0 - R_f^{-2} F_{02}$

absense of arbitrage implies that the Forward price satisfies $F_{02} = R_f^2 (S_0 - D_0)$ which implies that $E_0(m_{02} f_2) = 0$

2. **Assume that there is an economy populated by infinitely lived representative individuals who maximize the lifetime utility function**

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right]$$

where c_t is consumption at date t and $a>0$, $0<\delta<1$. The economy is a Lucas endowment economy (Lucas 1978) having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

Lucas model (1978): the price of risky asset, P_0 satisfy

$P_0 = E_0\left(\sum_{t=1}^{\infty} \frac{U_c(C_t,t)}{U_c(C_0,0)} d_t\right)$, $U(C_t,t) = -\delta^t e^{-aC_t}$. because this is an endowment economy with one share per individual, we have $C_t = d_t$. thus

$$P_0 = E_0\left(\sum_{t=1}^{\infty} \frac{U_c(C_t,t)}{U_c(C_0,0)} d_t\right) = E_0\left(\sum_{t=1}^{\infty} \delta^t e^{-\alpha(d_t - d_0)} d_t\right)$$

3. **For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).**

$$P_t = E_t\left(\sum_{t=1}^{\infty} \delta_i \left(\frac{C^*_{t+j}}{C^*_t}\right)^{r-1} d_{t+j}\right) \quad \text{or}$$

$$P_t/d_t = E_t\left(\sum_{j=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{r-1}\left(\frac{d_{t+j}}{d_t}\right)\right) = E_t\left(\sum_{j=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{r-1}\left(\frac{d_{t+j}}{d_t}\right)\right)$$

So $\ln(C_{t+j}/C_t) = j\cdot\mu_c + \sigma_c \sum_{j=1}^{j} n_{t+j}$

$\ln(D_{t+j}/D_t) = j\cdot\mu_d + \sigma \sum_{j=1}^{j} \varepsilon_{t+j}$

then $P_t/d_t = E_t\left(\sum_{j=1}^{\infty} \delta^j e^{(r-1)(j\mu_c + \sigma_c \sum_{j=1}^{j} n_{t+j}) + j\mu_c + \sigma_c \sum_{j=1}^{j} \varepsilon_{t+j}}\right)$

$$= E_t\left(\sum_{j=1}^{\infty} \delta^j e^{((r-1)\mu_d + \mu_c + \sum_{j=1}^{\infty}((r-1)\sigma_c n_{t+j} + \sigma_c \varepsilon_{t+j}))}\right)$$

$$= \sum_{j=1}^{\infty} \delta^j e^{((r-1)\mu_c + \sigma_c)} e^{r(((r-1)^2 \sigma_c^2 + \sigma_c^2) - (1-r)\sigma_c \sigma_d P)}$$

$$= \sum_{j=1}^{\infty} e^{(\ln\delta - (1-r)\mu_c + \frac{1}{2}((1-r)^2 \sigma_c^2 + \sigma_c^2) - (1-r)\sigma_c \sigma_d P)}$$

$$= \frac{1}{1-\delta e^{(1-r)\mu_c + \mu_c + \frac{1}{2}((1-r)^2\sigma_c^2 + \sigma_c^2) - (1-r)\sigma_c\sigma_d P}} - 1$$

So $P_t = d_t \dfrac{\delta e^{v}}{1-\delta e^{v}}$

where $\alpha = \mu_d - (1-r)\mu_c + \frac{1}{2}((1-r)^2 \sigma_c^2 + \sigma_d^2) - (1-r)P\sigma_c\sigma_d$

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t that is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t[d_{t+i}]}{R_f^{i+1}}1 \quad (1)$$

and

$$b_{t+1} = \begin{cases} \frac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1-q_t \end{cases} \quad (2)$$

where $E_t[e_{t+1}] = E_t[z_{t+1}] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

a. Show whether or not $p_t = f_t + b_t$, subject to the specifications in (1) and (2), is a valid solution for the price of the risky asset.

Check that (2) satisfies $E_t(b_{t+1}) = R_f b_t$, q: random variable as of date $t-1$ it is realized as date t, thus

$$E_t(b_{t+1}) = \frac{R_f}{q} b_t q_t + E_t(e_{t+1}) q_t + (1-q_t) E_t(z_{t+1}) = R_f b_t$$

the solution is valid

b. Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

Since $E_t(b_{t+1}) = R_f b_t$ we see that $\lim_{i \to \infty} E_t(b_{t+i}) = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$ (*)

For limited liability oil asset, we can't have a bubble path with negative price, then we need to consider only bubble with $b_t > 0$ In this case, from the above equation that for the bubble solution to exist the bubble components should be expected to increase infinity. but this can't be rational expectation if there's an upper bound on the price of oil, as would be the case if there's was a perfect substitute in perfect elastic supply So bubble path where b_t must be expected to increase to infinity cannot possible occur.

c. Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

This is also similar, a rational speculative bubble can't exist for the price of bond. Since at maturity, the bond price must be $P_t = d_t$ and zero after date T, its price can't rationally be expected to satisfy equation (*) and increase infinity.

Thus a bubble path is invalid, and the only rational price is $P_t = P_t^*$

5. **Consider an endowment economy with representative agents who maximize the following objective function:**

$$\max_{C_s(\omega_s),\forall s,t} E_t\left[\sum_{s=t}^{T}\delta^s u(C_s)\right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

With the economy and therefore asset, having a finite horizontal, asset price couldn't have the form $P_t = f_t + b_t$ with $b \neq 0$ because out date T, $P_T = f_T = d_T$ which is an asset's final dividend payment. Since $b_T = 0$ which certainty, then the bubble process
$E_t(b_{t+1}) = \delta^{-1} b_t$ implies $E_{T-1}(b_T) = E_{T-1}(0) = \delta^{-1} b_{T-1}$ or $b_{T-1} = 0$.
Similar argument implies $b_t = 0$ for all previous date, $t < T-1$