Applicatiosns of Derivatives
Part 1: L'Hôpital's Rule and Related Rate

1 L'Hôpital's Rule

This section uses derivative to calculate certain limits with indeterminate forms.

Definition 1.1 (Indeterminate Form)**.** The limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

has the **indeterminate form** $\frac{0}{0}$ **at** $x = a$ if

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0,$$

and has the **indeterminate form** $\frac{\infty}{\infty}$ **at** $x = a$ if

$$\lim_{x\to a} |f(x)| = \infty \qquad \text{and} \qquad \lim_{x\to a} |g(x)| = \infty,$$

i.e. either (i) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ or
(ii) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = -\infty$, or
(iii) $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to a} g(x) = +\infty$, or
(iv) $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to a} g(x) = -\infty$.

Note that the definition for indeterminate form above also holds for the one-sided limit and the limit at infinity:

$$x \to a^-, \qquad x \to a^+, \qquad x \to -\infty, \qquad x \to \infty.$$

Theorem 1.1 (L'Hôpital's Rule)**.** Suppose f and g are differentiable functions on an open interval containing the number a, except a itself, and that $g'(x) \neq 0$ for all x in the interval, except possibly at a.
Suppose $\lim_{x\to a} f(x)/g(x)$ is an indeterminate form and $\lim_{x\to a} f'(x)/g'(x) = L$ or $\pm\infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example 1.1. Evaluate $\lim_{x\to a} \frac{\sin(x)}{x}$.

This limit has the indeterminate form 0/0 at $x = 0$, because

So we can apply L'Hôpital's rule as follows.

Example 1.2. Evaluate $\lim_{x\to 0} \frac{\sin(x)}{e^x - e^{-x}}$. [Ans: 1/2]

Solution: This limit has the indeterminate form $\frac{0}{0}$ at $x = 0$. So we can apply L'Hôpital's rule as follows.

Example 1.3. Evaluate $\lim_{x\to \infty} \frac{\ln(x)}{e^x}$. [Ans: 0]

Solution: This limit has the indeterminate form $\frac{\infty}{\infty}$, because

So we can apply L'Hôpital's rule as follows.

Example 1.4. Evaluate $\lim_{x\to \infty} \frac{6x^2+5x+7}{4x^2+2x}$ by applying L'Hôpital's rule. [Ans: 0]

Solution: This limit has the indeterminate form $\frac{\infty}{\infty}$. This problem has to successively apply L'Hôpital's rule.

Example 1.5. Evaluate $\lim_{t\to\frac{\pi}{2}^+} \frac{\tan(t)}{\tan(3t)}$. [Ans: 3]

Solution: This problem has to successively apply L'Hôpital's rule. This given limit has the indeterminate form $\frac{\infty}{\infty}$ because
[Hint: Use $\sin(2A) = 2\sin(A)\cos(A)$]

Example 1.6 (Exercise)**.** Evaluate $\lim_{x\to 1^+} \frac{\ln(x)}{\sqrt{x-1}}$. [Ans: 0]
Solution: This limit has the indeterminate form

1.1 Other Indeterminate Forms

We will consider additional indeterminate forms:

$$\infty - \infty, \qquad 0 \cdot \infty, \qquad 0^0, \qquad \infty^0, \qquad 1^\infty,$$

which **can be rearranged to be in the indeterminate form, either in $\frac{0}{0}$ or $\frac{\infty}{\infty}$**.

Example 1.7. Evaluate $\lim_{x\to 0^+}\left[\frac{3x+1}{\sin(x)} - \frac{1}{x}\right]$. [Ans: 3]
Solution: This limit has the indeterminate form
Rearranging by using the common denominator will give the indeterminate form $\frac{0}{0}$.

Example 1.8 (Exercise)**.** Evaluate $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$. [Ans: 1]
Solution: This limit has the indeterminate form $0 \cdot \infty$. Notice that we can write

$$x\sin\left(\frac{1}{x}\right) = \frac{\sin(1/x)}{1/x}.$$

The indeterminate forms: 0^0, ∞^0, 1^∞

Suppose $\lim_{x\to a} f(x)^{g(x)}$ is in the form 0^0, ∞^0 or 1^∞. Let $\boxed{y = f(x)^{g(x)}}$.

$$\begin{aligned}\ln(y) &= \ln(f(x)^{g(x)})\\ \ln(y) &= g(x)\ln(f(x))\\ \lim_{x\to a}\ln(y) &= \lim_{x\to a} g(x)\ln(f(x))\end{aligned}$$

will be in the indeterminate form $0\cdot\infty$ (why?). Suppose the L'Hôpital's rule is applied to obtain

$$L = \lim_{x\to a} g(x)\ln(f(x))$$

and using the continuity of function $\ln(\cdot)$:

$$\ln\left(\lim_{x\to a} y\right) = \lim_{x\to a}\ln(y) = \lim_{x\to a} g(x)\ln(f(x)) = L.$$

That is,

$$\boxed{\ln\left(\lim_{x\to a} y\right) = L \qquad \Rightarrow \qquad \lim_{x\to a} y = e^L.}$$

Note that the process above also holds for the one-sided limit and the limit at infinity:

$$x\to a^-, \qquad x\to a^+, \qquad x\to -\infty, \qquad x\to\infty.$$

Example 1.9. Evaluate $\lim_{x\to 0} x^{1/\ln(x)}$

[Ans: e]

Solution:

Note that $\lim_{x\to 0}\ln(x) = -\infty$. So $\lim_{x\to 0} x^{1/\ln(x)}$ is in the form 0^0.

Example 1.10 (Exercise)**.** Evaluate $\lim_{x\to\infty} \left(1 - \frac{3}{x}\right)^{2x}$
[Ans: e^{-6}]
Solution: Note that this is in the form 1^0.

2 Related Rate

The derivative $\frac{dy}{dx}$ of a function $y = f(x)$ is its instantaneous rate of change with respect to variable x.

- Suppose a function $s(t)$ represent the position o an object moving on a horizontal or vertical line. Then the **time rate of change** $\frac{ds}{dt}$ is the velocity of the object.
- In general, a time rate of change tells how fast the quantity is changing.

Guidelines for Solving Related Problems

(i) Carefully read the problem. Draw a picture if possible.

(ii) Label with symbols all quantities that change with time.

(iii) Write down all the rates that are given. Using derivative notation, write down the rate that you want to find.

(iv) Set up an equation or a function that relates all the variables you have introduced.

(v) Differentiate the equation or function found in step (iv) with respect to t. This may need to use **implicit differentiation**. The resulting equation after differentiation rates at which the variables change with time.

Example 2.1. Air is being pumped into a spherical balloon at a rate of 20 ft^3/min. At what rate is the radius changing when the radius is 3 ft ?

[Ans: $\frac{5}{9\pi}$ ft/min]
Solution:

Example 2.2. A woman jogging at a constant rate of 10 km/h crosses a point P heading north. Ten minutes later a man jogging at a constant rate of 9 km/h crosses the same point heading east. How fast is the distance between the joggers changing 20 minutes after the man crosses P ? [Ans: $\frac{77}{\sqrt{34}}$ km/hr]

Solution: Let x and y be the locations of the man and the women from point P, respectively. Let z be the distance between the man and the woman. Then by the **Pythagorean theorem**, $\boxed{z^2 = x^2 + y^2.}$

Example 2.3. A light is on the top of a 12 ft tall pole and a 5.5ft tall person is walking away from the pole at a rate of 2 ft/sec.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

[Ans: 48/13, 22/13 ft/sec]
Solution:
Let x be the distance of the tip of the shadow from the pole,
x_p be the distance of the person from the pole and
x_s be the length of the shadow.

3 Additional Exercise

Example 3.1. A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec. Find the necessary rate of change of the cameras angle as a function of time so that it stays focused on the rocket.

Example 3.2. A cone-shaped (conical) tank is leaking water at a constant rate of 2 ft^3/hour. The base radius of the tank is 5 ft and the height of the tank is 14 ft.
(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?
(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft

Example 3.3. A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of 1/4 ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?