Question 0:

Consider the function f defined for all (x,y) such that

$$f\,(x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1 - a^3, a^2)$ is a stationary point of $f\,(x, y; a)$.
b. State the condition under which the above stationary point is a global maximum.
c. Given that $G(a) = f\,(x^*, y^*; a)$, show the derivative of G with respect to a.
d. Calculate $\frac{\partial f\,(x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1 - a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?
e. Determine the domain of (x,y) in the xy-plan where f is convex. $\partial^2 f > 0$

a.)

$$\frac{\partial f}{\partial x} = x - 1 + ay = 0$$

$$x = 1 - ay$$

$$x^* = 1 - a^3$$

$$\frac{\partial f}{\partial y} = ax - a - y^2 + 2a^2y = 0$$

$\boxed{\text{plug } x = 1-ay}$ $a(1-ay) - a - y^2 + 2a^2y = 0$

$$\cancel{a} - a^2y - \cancel{a} - y^2 + 2a^2y = 0$$

$$a^2y - y^2 = 0$$

$$a^2y = y^2$$

$$y^* = a^2$$

b)

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & -2y + 2a^2 \end{bmatrix}$$

$$|H_1| = 1$$

$$|H_2| = -2y + 2a^2 - a^2$$

$$= -2y + a^2$$

There is no global maximum

C.

$$G = \frac{1}{2}(1-a^3) - (1-a^3) + \left[(a)(a^2)(1-a^3-1)\right] - \frac{1}{3}(a^2) + a^2(a^2)$$

$$= \frac{1}{2} - \frac{1}{2}a^3 - 1 + a^3 - a^6 - \frac{1}{3}a^2 + a^4 = -\frac{1}{2} - \frac{1}{3}a^2 + \frac{1}{2}a^3 + a^4 - a^6$$

$$\frac{\partial G}{\partial a} = \frac{\partial f(x^*, y^*; a)}{\partial a} = -\frac{2}{3}a + \frac{3}{2}a^2 + 4a^3 - 6a^5$$

D.

$$\frac{\partial f(x, y; a)}{\partial a} = yx - y$$

plug $(x^*, y^*) = (1-a^3, a^2)$

$$= a^2 - a^5 - a^2$$

$$= a^5$$

answer in $C \neq D$

e.

e. Determine the domain of (x,y) in the xy-plan where f is convex. $\partial^2 f > 0$

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & -2y+2a^2 \end{bmatrix}$$

$$|H_1| = 1 > 0$$

$$|H_2| = -2y + 2a^2 - a^2$$

$$= -2y + a^2 > 0$$

$$a^2 > 2y$$

$$\frac{a^2}{2} > y$$

$X = (-\infty, \infty)$

$Y = (-\infty, \frac{a^2}{2})$

Question 2: Suppose that there are three groups of people who take Sky train to commute in Bangkok. The first group is students (s), the second group is senior citizens (old), and the third group is working-aged people. The demand for each group is given by the following equations:

Demand of students: $P_s = 8 - \left(\frac{1}{2}\right) Q_s$

Demand of senior citizens: $P_{old} = 16 - 2Q_{old}$

Demand of working-aged people: $P_w = 20 - Q_w$

The Sky train operator has a constant marginal cost at MC = \$4, and total cost at TC = 4Q + 10. Consider the following problems.

a) Determine the profit-maximizing level of output/price under third-degree price discrimination. Calculate the level of maximized profit.

b) Confirm your result in (a) with the second-order derivative test.

c) Calculate (i) consumer surplus for each of the three groups of consumers, and (ii) producer surplus.

d) Calculate the optimal level of total output if the Sky train operator can practice the first-degree price discrimination for each group of the consumers in the market.

a) $\pi = TR - TC$

$= P_s \cdot Q_s + P_{old} \cdot Q_{old} + P_w \cdot Q_w - (4(Q_s + Q_{old} + Q_w) + 10)$

$= (8 - \frac{1}{2}Q_s)Q_s + (16 - 2Q_{old})Q_{old} + (20 - Q_w)Q_w - (4(Q_s + Q_{old} + Q_w) + 10)$

$= 8Q_s - \frac{1}{2}Q_s^2 + 16Q_{old} - 2Q_{old}^2 + 20Q_w - Q_w^2 - 4Q_s - 4Q_{old} - 4Q_w - 10$

F.O.C

$$\frac{\partial \pi}{\partial Q_s} = 8 - Q_s - 4 = 0$$

$$Q_s^* = 4$$

$$\frac{\partial \pi}{\partial Q_{old}} = 16 - 4Q_{old} - 4 = 0$$

$$Q_{old}^* = 3$$

$$\frac{\partial \pi}{\partial Q_w} = 20 - 2Q_w - 4 = 0$$

$$Q_w^* = 8$$

b)

$$\begin{bmatrix} \pi_{ss} & \pi_{s\,old} & \pi_{sw} \\ \pi_{old\,s} & \pi_{old\,old} & \pi_{old\,w} \\ \pi_{ws} & \pi_{w\,old} & \pi_{ww} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} -1 & 0 \\ 0 & -4 \\ 0 & -2 \end{matrix}$$

$|H_1| = -1$

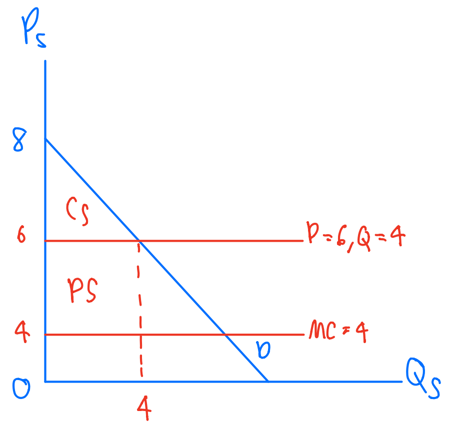
$|H_2| = (-1)(-4) - 0 = 4$

$|H_3| = (-8) - 0 = -8$

It is negative definite so it globally concave and it maximum.

c) Consumer surplus and producer surplus.

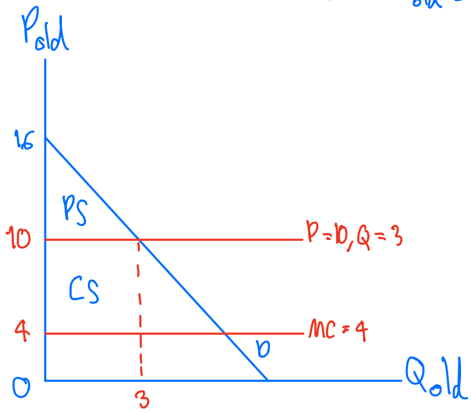
- Demand of student : $P_s = 8 - \frac{1}{2} Q_s$



$CS = \frac{1}{2}(8-6)(4-0) = \$4$

$PS = (6-4)(4) = \$8$

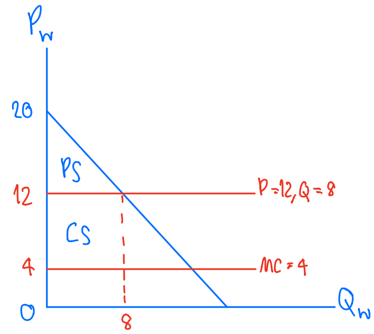
- Demand of senior citizens : $P_{old} = 16 - 2Q_{old}$



$CS = \frac{1}{2}(16-10)(3-0) = \$9$

$PS = (10-4)(3) = \$18$

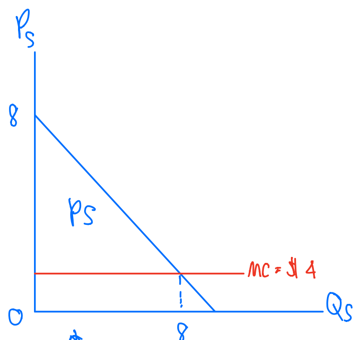
- Demand of working-aged people : $P_w = 20 - Q_w$



$CS = \frac{1}{2}(20-12)(8-0) = \$32$

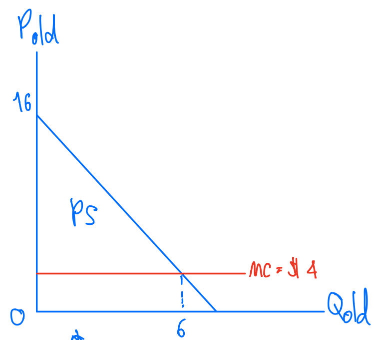
$PS = (12-4)(8) = \$64$

D) Student.



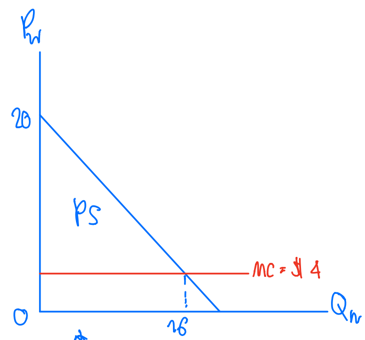
$Q^* = 8$ ※

Senior citizens



$Q^* = 6$ ※

Working-aged people



$Q^* = 16$ ※

Q.4 Suppose that the output Q of a firm depends on two inputs of the quantities K and L. The output level is determined by the production function

$$Q = 36K + 16L - 3K^2 - 2KL - L^2$$

a. Is the firm's production function strictly concave? Explain.

$$H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LK} & Q_{LL} \end{bmatrix}$$

$$H = \begin{bmatrix} -6 & -2 \\ -2 & -2 \end{bmatrix}$$

$$\frac{\partial Q}{\partial K} = 36 - 6K - 2L$$

$$\frac{\partial Q}{\partial L} = 16 - 2K - 2L$$

$|H_1| = -6$; $\forall K, \forall L \longrightarrow |H_1| < 0$

$|H_2| = (-6)(-2) - (-2)(-2) = 8$; $\forall K, \forall L \rightarrow |H_2| > 0$

$\therefore$ For any value of K and L, $|H_1| < 0$ and $|H_2| > 0$. $d^2Q < 0$, the firm's production function is strictly concave.

b. Determine the optimal input (K*, L*) that maximizes the output level.

$$\frac{\partial Q}{\partial K} = 36 - 6K - 2L = 0$$

$$L = 18 - 3K \quad \text{——} \ (1)$$

$$\frac{\partial Q}{\partial L} = 16 - 2K - 2L = 0$$

$$L = 8 - K \quad \text{——} \ (2)$$

$(1) = (2)$

$18 - 3k = 8 - k$

$2k = 10$

$k^* = 5$ #

Plug $K^* = 5$ in (2)

$L^* = 8 - 5 = 3$ #

c. Write down the firm's profit function when the price of Q is P and the per-unit factor prices of K and L are r and w, respectively, where both r and w are positive numbers. Find the levels of K^* and L^* that maximize the firm's profits.

Profit function = Total Revenue - Total Cost

$$\pi = P \cdot Q - rK - wL$$

$$\pi = P \cdot (36K + 16L - 3K^2 - 2KL - L^2) - rK - wL$$

Profit Maximising input factors

$$\frac{\partial \pi}{\partial k} = 36P - 6KP - 2LP - r = 0$$

$$P(36 - 6K - 2L) = r$$

$$\frac{36 - 6K - \frac{r}{P}}{2} = L$$

$$L = 18 - 3K - \frac{r}{2P} \quad \text{—①}$$

$$\frac{\partial \pi}{\partial L} = 16P - 2KP - 2LP - W = 0$$

$$16 - 2K - 2L = \frac{W}{P}$$

$$L = 8 - K - \frac{W}{2P} \quad \text{—②}$$

① = ②

$$18 - 3K - \frac{r}{2P} = 8 - K - \frac{W}{2P}$$

$$10 - \frac{r}{2P} + \frac{W}{2P} = 2k$$

$$K^* = 5 - \frac{r}{4P} + \frac{W}{4P} \quad \#$$

Plug K^* into ②

$$L = 8 - \left(5 - \frac{r}{4P} + \frac{W}{4P}\right) - \frac{W}{2P}$$

$$L^* = 3 + \frac{r}{4P} - \frac{W}{4P} - \frac{W}{2P}$$

$$L^* = 3 + \frac{r}{4P} - \frac{W}{4P} \quad \#$$

d. Verify that the second-order sufficient conditions for maximum profits are satisfied.

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} -6P & -2P \\ -2P & -2P \end{bmatrix}$$

$$|H_1| = -6P \ ; \forall K, \forall L \longrightarrow |H_1| < 0$$

$$|H_2| = (-6P)(-2P) - (-2P)(-2P) = 8P^2 \ ; \forall K, \forall L \longrightarrow |H_2| > 0$$

∴ For any value of K and L, $|H_1| < 0$ and $|H_2| > 0$. $d^2Q < 0$, K^* and L^* are the profit maximisers

e. Determine the effect of an increase in *r* on the firm's use of each input. (i.e. determine $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial r}$).

$K^* = 5 - \frac{r}{4P} + \frac{w}{4P}$; $L^* = 3 + \frac{r}{4P} - \frac{w}{4P}$

$$\frac{\partial K^*}{\partial r} = -\frac{1}{4P} \ \#$$

When the cost of capital increases by 1 unit, the optimal K^* falls by $\frac{1}{4P}$ unit.

$$\frac{\partial L^*}{\partial r} = \frac{1}{4P}$$

When the cost of capital increases by 1 unit, the optimal L^* falls by $\frac{1}{4P}$ unit.