Appendix 9A

The Basics of Bond Valuation

BOND VALUATION

The valuation of a bond instrument employs time value of money concepts. The fair value of a bond reflects the present value of all cash flows promised or projected to be received on that bond discounted at the required rate of return (*rrr*). Similarly, the expected rate of return (*Err*) is the interest rate that equates the current market price of the bond with the present value of all promised cash flows received over the life of the bond. Finally, a realized rate of return (*rr*) on a bond is the actual return earned on a bond investment that has already taken place. Promised cash flows on bonds come from two sources: (1) interest or coupon payments paid over the life of the bond and (2) a lump-sum payment (face or par value) when a bond matures.

Bond Valuation Formula Used to Calculate Fair Present Values

coupon bonds
Bonds that pay interest based on a stated coupon rate. The interest paid or coupon payment per year is generally constant over the life of the bond.

zero-coupon bonds
Bonds that do not pay interest.

Most bonds pay a stated coupon rate of interest to the holders of the bonds. These bonds are called **coupon bonds.** The interest, or coupon, payments per year are generally constant (fixed) over the life of the bond.[1] Thus, the fixed interest payment, C, is essentially an annuity paid to the bondholder periodically (normally semiannually) over the life of the bond. Bonds that do not pay coupon interest are called **zero-coupon,** or discount, **bonds.** For these bonds C is zero. The face or par value of the bond, on the other hand, is a lump-sum payment received by the bondholder when the bond matures. Face value is generally set at $1,000 in the U.S. bond market.

Using time value of money formulas, and assuming that the bond issuer makes its promised coupon and principal payments, the present value of a bond can be written as

$$\begin{aligned} P &= \frac{C/m}{(1+i_d/m)^1} + \frac{C/m}{(1+i_d/m)^2} + \cdots + \frac{C/m}{(1+i_d/m)^{Nm}} + \frac{F}{(1+i_d/m)^{Nm}} \\ &= \frac{C}{m}\sum_{t=1}^{Nm}\left(\frac{1}{1+i_d/m}\right)^t + \frac{F}{(1+i_d/m)^{Nm}} \\ &= \frac{C}{m}(PVA_{i_d/m,Nm}) + F(PV_{i_d/m,Nm}) \end{aligned}$$

where

P = Present value of the bond

F = Par or face value of the bond

C = Annual interest (or coupon) payment per year on the bond; equals the par value of the bond times the (percentage) coupon rate

[1] Variable rate bonds pay interest that is indexed to some broad interest rate measure (such as Treasury bills) and thus experience variable coupon payments. Income bonds pay interest only if the issuer has sufficient earnings to make the promised payments. Index (or purchasing power) bonds pay interest based on an inflation index. Each of these types of bonds, therefore, can have variable interest payments.

N = Number of years until the bond matures

m = Number of times per year interest is paid

i_d = Interest rate used to discount cash flows on the bond

EXAMPLE 9A–1
Calculation of the Fair Value of a Coupon Bond

You are considering the purchase of a bond that pays 10 percent coupon interest per year, with the coupon paid semiannually (i.e., 5 percent over the first half of the year and 5 percent over the second half of the year). The bond matures in 12 years and has a face value of $1,000. If the required rate of return (*rrr*) on this bond is 8 percent, the fair market value of the bond is calculated as follows:

$$P = \frac{1{,}000(0.1)}{2}(PVA_{8\%/2,12(2)}) + 1{,}000(PV_{8\%/2,12(2)})$$
$$= 50(15.24696) + 1{,}000(0.39012) = \$1{,}152.47$$

or an investor would be willing to pay no more than $1,152.47 for this bond.[2]

If the required rate of return on this bond is 10 percent, the fair market value of the bond is calculated as follows:

$$P = \frac{1{,}000(0.1)}{2}(PVA_{10\%/2,12(2)}) + 1{,}000(PV_{10\%/2,12(2)})$$
$$= 50(13.79864) + 1{,}000(0.31007) = \$1{,}000.00$$

or an investor would be willing to pay no more than $1,000.00 for this bond.

If the required rate of return on this bond is 12 percent, the fair market value of the bond is calculated as follows:

$$P = \frac{1{,}000(0.1)}{2}(PVA_{12\%/2,12(2)}) + 1{,}000(PV_{12\%/2,12(2)}) = \$874.50$$
$$= 50(12.55036) + 1{,}000(0.24698) = \$874.50$$

or an investor would be willing to pay no more than $874.50 for this bond.

premium bond
A bond in which the present value of the bond is greater than its face value.

discount bond
A bond in which the present value of the bond is less than its face value.

In the preceding example, notice that when the required rate of return (*rrr*) on the bond is 8 percent, the fair value of the bond, $1,152.47, is greater than its face value of $1,000. When this relationship between the fair value and the face value of a bond exists, the bond is referred to as a **bond** that should sell at a **premium.** This premium occurs because the coupon rate on the bond is greater than the required rate of return on the bond (a 10 percent coupon rate versus an 8 percent required rate of return in our example). When the required rate of return on the bond is 12 percent, the present value of the bond is less than its face value, and the bond is referred to as a **bond** that should sell at a **discount.** This discount occurs because the coupon rate on the bond is less than the required rate of return on the bond. Finally, when the required rate of return on the bond is 10 percent, the

[2] If the bond paid interest once per year (i.e., $m = 1$) rather than twice, the bond's fair market value is calculated as

$$P = 1{,}000(0.1)(PVA_{8\%,12}) + 1{,}000(PV_{8\%,12}) = \$1{,}150.72$$

TABLE 9A–1
Description of a Premium, Discount, and Par Bond

Premium Bond—when the *coupon rate* on a bond is greater than the *required rate of return* on the bond, the *fair present value* is greater than the *face value* of the bond. When the *coupon rate* on a bond is greater than the *yield to maturity* on the bond, the *current market price* is greater than the face value of the bond. **Discount Bond**—when the *coupon rate* on a bond is less than the *required rate of return* on the bond, the *fair present value* is less than the *face value of the bond.* When the coupon rate on a bond is less than the *yield to maturity* on the bond, the *current market price* is less than the *face value* of the bond. **Par Value**—when the *coupon rate* on a bond is equal to the *required rate of return* on the bond, the *fair present value* is equal to the *face value* of the bond. When the coupon rate on a bond is equal to the *yield to maturity* on the bond, the *current market price* is equal to the *face value* of the bond.

par bond
A bond in which the present value of the bond is equal to its face value.

present value of the bond is equal to its face value, and the bond is referred to as a **bond** that should sell at **par.** This par occurs because the coupon rate on the bond is equal to the required rate of return on the bond. To achieve the required rate of return on the bond, the bondholder experiences neither a gain nor a loss on the difference between the purchase price of the bond and the face value received at maturity. We summarize the scenarios for premium, discount, and par bonds in Table 9A–1.

It should be noted that the designation as a premium, discount, or par bond does not necessarily assist a bondholder in the decision to buy or sell a bond. This decision is made on the basis of the relationship between the fair present value and the actual current market price of the bond. Rather, premium, discount, and par bonds are descriptive designations regarding the relationship between the fair present value of the bond and its face value. The fair present value of the bond will only equal the bond's current market price in an efficient market where prices instantaneously adjust to new information about the security's value.

Bond Valuation Formula Used to Calculate Yield to Maturity

yield to maturity
The return or yield the bondholder will earn on the bond if he or she buys it at its current market price, receives all coupon and principal payments as promised, and holds the bond until maturity.

The present value formulas also can be used to find the expected rate of return (*Err*) or, assuming all promised coupon and principal payments are made with a probability of 100 percent, what is often called the **yield to maturity** (*ytm*) on a bond (i.e., the return the bondholder would earn on the bond if he or she buys the bond at its current market price, receives all coupon and principal payments as promised, and holds the bond until maturity). The yield to maturity calculation implicitly assumes that all coupon payments periodically received by the bondholder can be reinvested at the same rate—that is, reinvested at the calculated yield to maturity.

Rewriting the bond valuation formula, where P is the current market price that has to be paid to buy the bond, we can solve for the yield to maturity (*ytm*) on a bond as follows (where we write *ytm* instead of *Err*):

$$P = \frac{C/m}{(1 + ytm/m)^1} + \frac{C/m}{(1 + ytm/m)^2} + \cdots + \frac{C/m}{(1 + ytm/m)^{Nm}} + \frac{F}{(1 + ytm/m)^{Nm}}$$

$$= \frac{C}{m}(PVA_{ytm/m,Nm}) + F(PV_{ytm/m,Nm})$$

TABLE 9A–2
Summary of Factors That Affect Security Prices and Price Volatility When Interest Rates Change

Interest Rate—there is a negative relation between interest rate changes and present value (or price) changes on financial securities.
As interest rates increase, security prices decrease at a decreasing rate.
Time Remaining to Maturity—the shorter the time to maturity for a security, the closer the price is to the face value of the security.
The longer the time to maturity for a security, the larger the price change of the security for a given interest rate change.
The maturity effect described above increases at a decreasing rate.
Coupon Rate—the higher a security's coupon rate, the smaller the price change on the security for a given change in interest rates.

EXAMPLE 9A–2
Calculation of the Yield to Maturity on a Coupon Bond

You are considering the purchase of a bond that pays 11 percent coupon interest per year, paid semiannually (i.e., 5½ percent per semiannual period). The bond matures in 15 years and has a face value of $1,000. If the current market price of the bond is $931.176, the yield to maturity (or *Err*) is calculated as follows:

$$931.176 = 1{,}000(0.11)/2\ (PVA_{ytm/2,15(2)}) + 1{,}000(PV_{ytm/2,15(2)})$$

Solving for *ytm*, the yield to maturity (or expected rate of return) on the bond is 12 percent.[3] Equivalently, you would be willing to buy the bond only if the required rate of return (*rrr*) was no more than 12 percent.

The variability of financial security prices depends on interest rates and the characteristics of the security. Specifically, the factors that affect financial security prices include interest rate changes, the time remaining to maturity, and the coupon rate. We evaluate next the impact of each of these factors as they affect bond prices. Table 9A–2 summarizes the major relationships we will be discussing.

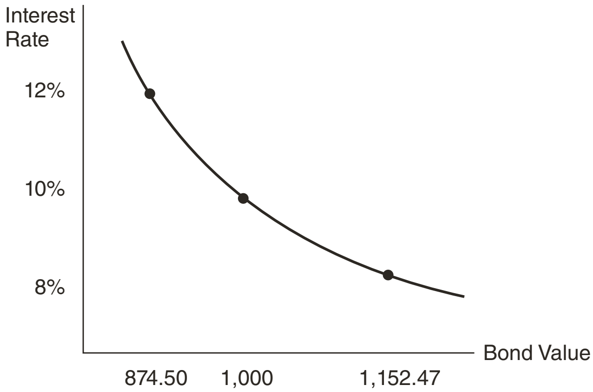
IMPACT OF INTEREST RATE CHANGES ON SECURITY VALUES

Refer back to Example 9A–1. Notice in this example that present values of the cash flows on bonds decreased as interest rates increased. Specifically, when the required rate of return increased from 8 percent to 10 percent, the fair present value of the bond fell from $1,152.47 to $1,000, or by 13.23 percent ((1,000 − 1,152.47)/1,152.47). Similarly, when the required rate of return increased from 10 percent to 12 percent, the fair present value of the bond fell from $1,000 to $874.50, or by 12.55 percent ((874.50 − 1,000)/1,000). This is the inverse relationship between present values and interest rates. While the examples refer to the relation between fair values and required rates of returns, the inverse relation also exists between current market prices and expected rates of return—as yields on bonds increase, the current market prices of bonds decrease. We illustrate this inverse relation between interest rates on bonds and the present value of bonds in Figure 9A–1.

[3] The yield to maturity is the nominal return on the bond. Its effective annual return is calculated as

$$EAR = [1 + ytm/m]^m - 1 = [1 + 0.12/2]^2 - 1 = 12.36\%$$

FIGURE 9A–1
Relation between Interest Rates and Bond Values



Notice too from the earlier example that the inverse relationship between bond prices and interest rates is not linear. Rather, the percentage change in the present value of a bond to a given change in interest rates is smaller when interest rates are higher. When the required rate of return on the bond increased from 8 percent to 10 percent (a 2 percent increase), the fair present value on the bond decreased by 13.23 percent. However, another 2 percent increase in the required rate of return (from 10 percent to 12 percent) resulted in a fair present value decrease of only 12.55 percent. The same nonlinear relation exists for current market prices and yield to maturities. Thus, as interest rates increase, present values of bonds (and bond prices) decrease at a decreasing rate. This is illustrated in Figure 9A–1.

The relationship between interest rates and security values is important for all types of investors. Financial institutions (FIs) such as commercial banks, thrifts, and insurance companies are affected because the vast majority of the assets and liabilities held by these firms are financial securities (e.g., loans, deposits, investment securities). When required rates of return rise (fall) on these securities, the fair present values of the FI's asset and liability portfolios decrease (increase) by possibly different amounts, which in turn affects the fair present value of the FI's equity (the difference between the fair present value of an FI's assets and liabilities).

For example, suppose an FI held the 8 percent required return bond evaluated in Example 9A–1 (10 percent coupon interest per year paid semiannually, 12 years remaining to maturity, and face value of $1,000) in its asset portfolio and had partly financed the asset purchase by issuing the 10 percent required return bond evaluated in Example 9A–1 (the same bond characteristics as above except that the required rate of return is 10 percent) as part of its liability portfolio. In the example, we calculated the fair present values of these bonds as $1,152.47 and $1,000, respectively. The market value balance sheet of the FI is shown in Table 9A–3. The market value of the FI's equity is $152.47 ($1,152.47 − $1,000)—the difference between the market values of the FI's assets and liabilities. This also can be thought of as the value of the FI's equity owners' contribution to the financing of the purchase of the asset. If the required rate of return increases by 2 percent on both of these bonds (to 10 percent on the bond in the asset portfolio and to 12 percent on the bond in the liability portfolio), the fair present values of the asset and liability portfolios fall to $1,000 and $874.50, respectively. As a result, the value of the FI's equity falls to $125.50 ($1,000 − $874.50)—see Table 9A–3. Implicitly, the equity owners of the FI have lost $26.97 ($152.47 − $125.50) of the value of their ownership stake in the FI.

TABLE 9A–3
Balance Sheet of an FI before and after an Interest Rate Increase

(a) Balance Sheet before the Interest Rate Increase

Assets		**Liabilities and Equity**	
Bond (8% required rate of return)	$1,152.47	Bond (10% required rate of return)	$1,000
		Equity	$152.47

(b) Balance Sheet after 2 Percent Increase in the Interest Rates

Assets		**Liabilities and Equity**	
Bond (10% required rate of return)	$1,000	Bond (12% required rate of return)	$874.50
		Equity	$125.50

IMPACT OF MATURITY ON SECURITY VALUES

price sensitivity
The percentage change in a bond's present value for a given change in interest rates.

An important factor that affects the degree to which the price of a bond changes (or the price sensitivity of a bond changes) as interest rates change is the time remaining to maturity on the bond. A bond's **price sensitivity** is measured by the percentage change in its present value for a given change in interest rates. The larger the percentage change in the bond's value for a given interest rate change, the larger the bond's price sensitivity. Specifically, as is explained below, the shorter the time remaining to maturity, the closer a bond's price is to its face value. Also, the further a bond is from maturity, the more sensitive the price (fair or current) of the bond as interest rates change. Finally, the relationship between bond price sensitivity and maturity is not linear. As the time remaining to maturity on a bond increases, price sensitivity increases but at a decreasing rate. Table 9A–4 presents the bond information we will be using to illustrate these relationships.

Maturity and Security Prices

Table 9A–4 lists the present values of 10 percent (compounded semiannually) coupon bonds with a $1,000 face value and 12 years, 14 years, and 16 years, respectively, remaining to maturity. We calculate the fair present value of these bonds using an 8 percent, 10 percent, and 12 percent required rate of return. Notice that for each of these bonds, the closer the bond is to maturity, the closer the fair present value of the bond is to the $1,000 face value. This is true regardless of whether the bond is a premium, discount, or par bond. For example, at an 8 percent interest rate, the 12-year, 14-year, and 16-year bonds have present values of $1,152.47, $1,166.63, and $1,178.74, respectively. The intuition behind this is that nobody would pay much more than the face value of the bond and any remaining (in this case semiannual) coupon payments just prior to maturity since these are the only cash flows left to be paid on the bond. Thus, the time value effect is reduced as the maturity of the bond approaches. Many people call this effect the pull to par—bond prices and fair values approach their par values (e.g., $1,000) as time to maturity declines toward zero.

TABLE 9A–4 The Impact of Time to Maturity on the Relation between a Bond's Fair Present Value and Its Required Rate of Return

	12 Years to Maturity			14 Years to Maturity			16 Years to Maturity		
Required Rate of Return	Fair Price*	Price Change	Percentage Price Change	Fair Price*	Price Change	Percentage Price Change	Fair Price*	Price Change	Percentage Price Change
8%	$1,152.47			$1,166.63			$1,178.74		
		−$152.47	−13.23%		−$166.63	−14.28%		−$178.74	−15.16%
10%	1,000.00			1,000.00			1,000.00		
		−125.50	−12.55		−134.06	−13.41		−140.84	−14.08
12%	874.50			865.94			859.16		

* The bond pays 10% coupon interest compounded semiannually and has a face value of $1,000.

MATURITY AND SECURITY PRICE SENSITIVITY TO CHANGES IN INTEREST RATES

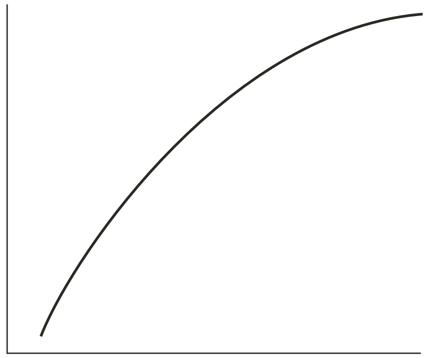
The Percentage Price Change columns in Table 9A–4 provide data to examine the effect time to maturity has on bond price sensitivity to interest rate changes. From these data we see that the longer the time remaining to maturity on a bond, the more sensitive are bond prices to a given change in interest rates. (Note again that all bonds in Table 9A–4 have a 10 percent coupon rate and a $1,000 face value.) For example, the fair present value of the 12-year bond falls 13.23 percent (i.e., ($1,000 − $1,152.47)/$1,152.47 = −0.1323 = −13.23%) as the required rate of return increases from 8 percent to 10 percent. The same 2 percent increase (from 8 percent to 10 percent) in the required rate of return produces a larger 14.28 percent drop in the fair present value of the 14-year bond, and the 16-year bond's fair present value drops 15.16 percent. This same trend is demonstrated when the required rate of return increases from 10 percent to 12 percent—the longer the bond's maturity, the greater the percentage decrease in the bond's fair present value.

The same relationship occurs when analyzing expected rates of return (or yields to maturity) and the current market price of the bond—the longer the time to maturity on a bond, the larger the change in the current market price of a bond for a given change in yield to maturity.

Incremental Changes in Maturity and Security Price Sensitivity to Changes in Interest Rates

A final relationship we can examine from Table 9A–4 is that between incremental changes in time remaining to maturity and incremental changes in security price sensitivity to a given change in interest rates. Specifically, notice that the maturity effect described above is not linear. For example, a 2 percent increase in the required rate of return (from 8 percent to 10 percent) on the 12-year bond produces a 13.23 percent (i.e., ($1,000 − $1,152.47)/$1,152.47 = −0.1323 = −13.23%) decrease in the bond's fair present value. The same 2 percent increase (from 8 percent to 10 percent) in the 14-year bond produces a 14.28 percent decrease in the fair present value. The difference, as we move from a 12-year to a 14-year maturity, is 1.05 percent (14.28% − 13.23%). Increasing the time to maturity two more years

FIGURE 9A–2
The Impact of a Bond's Maturity on Its Interest Rate Sensitivity



(from 14 years to 16 years) produces an increase in price sensitivity of 0.88 percent ($-14.28\% - (-15.16\%)$). While price sensitivity for a given increase in interest rates increases with maturity, the increase is nonlinear (decreasing) in maturity. We illustrate this relationship in Figure 9A–2.

IMPACT OF COUPON RATES ON SECURITY VALUES

Another factor that affects the degree to which the price sensitivity of a bond changes as interest rates change is the bond's coupon rate. Specifically, the higher the bond's coupon rate, the higher its present value at any given interest rate. Also, the higher the bond's coupon rate, the smaller the price changes on the bond for a given change in interest rates. These relationships hold when evaluating either required rates of return and the resulting fair present value of the bond or expected rates of return and the current market price of the bond. To understand these relationships better, consider again the bonds in Example 9A–1. Table 9A–5 summarizes the bond values and value changes as interest rates change.

Coupon Rate and Security Price

In Table 9A–5, we first list the fair present values of the bonds analyzed in Example 9A–1. We then repeat the present value calculations using a bond with identical characteristics except for the coupon rate: 10 percent versus 12 percent. Notice that the fair present value of the 10 percent coupon bond is lower than that of the 12 percent coupon bond at every required rate of return. For example, when the required rate of return is 8 percent, the fair value of the 10 percent coupon bond is $1,152.47 and that of the 12 percent coupon bond is $1,304.94.

Coupon Rate and Security Price Sensitivity to Changes in Interest Rates

Table 9A–5 also demonstrates the effect a bond's coupon rate has on its price sensitivity to a given change in interest rates. The intuition behind this relation is as follows. The higher (lower) the coupon rate on the bond, the larger (smaller) is the portion of the required rate of return paid to the bondholder in the form of coupon payments. Any security that returns a greater (smaller) proportion of an investment sooner is more (less) valuable and less (more) price volatile.

To see this, notice in Table 9A–5 that the higher the bond's coupon rate, the smaller the bond's price sensitivity for any given change in interest rates. For

TABLE 9A–5
The Impact of Coupon Rate on the Relation between a Bond's Fair Present Value and Its Required Rate of Return

	10 Percent Coupon Bond			12 Percent Coupon Bond		
Required Rate of Return	Fair Price*	Price Change	Percentage Price Change	Fair Price*	Price Change	Percentage Price Change
8%	$1,152.47			$1,304.94		
		−$152.47	−13.23%		−$166.95	−12.79%
10%	1,000.00			1,137.99		
		−125.50	−12.55		−137.99	−12.13
12%	874.50			1,000.00		

* The bond pays interest semiannually, has 12 years remaining to maturity, and has a face value of $1,000.

example, for the 10 percent coupon bond, a 2 percent increase in the required rate of return (from 8 percent to 10 percent) results in a 13.23 percent decrease in the bond's fair price. A further 2 percent increase in the required rate of return (from 10 percent to 12 percent) results in a smaller 12.55 percent decrease in the fair price.

For the 12 percent coupon bond, notice that the 2 percent increase in the required rate of return (from 8 percent to 10 percent) results in a 12.79 percent decrease in the bond's fair price, while an increase in the required rate of return from 10 percent to 12 percent results in a smaller 12.13 percent decrease in the bond's fair price. Thus, price sensitivity on a bond is negatively related to the level of the coupon rate on a bond. The higher the coupon rate on the bond, the smaller the decrease in the bond's fair price for a given increase in the required rate of return on the bond.

We illustrate this relationship in Figure 9A–3. The high coupon-paying bond is less susceptible to interest rate changes than the low coupon-paying bond. This is represented in Figure 9A–3 by the slope of the line representing the relation between interest rates and bond prices. The sensitivity of bond prices is smaller (the slope of the line is flatter) for high-coupon bonds than for low-coupon bonds.

FIGURE 9A–3
The Impact of a Bond's Coupon Rate on Its Interest Rate Sensitivity