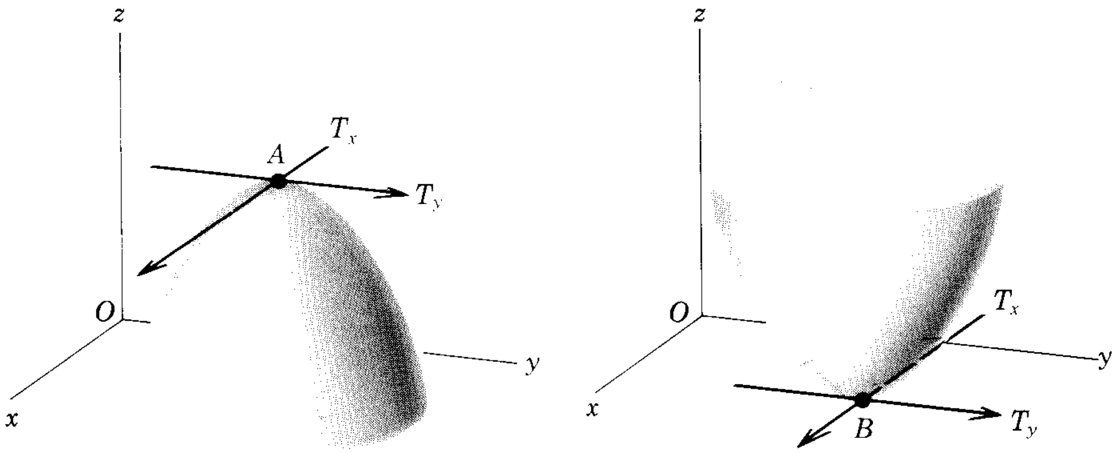
EE320 (1/2011)

Chapter 8

Optimization without Constraints: More-Than-One Independent Variable Cases

8.1 Optimization and Derivative



(a) Maximum point

(b) Minimum point

Let $y = f(x_1, x_2)$

Optimization:

First-Order condition: $f'(x_1) = 0$

$f'(x_2) = 0$

Second-Order condition:

Case I:

If $f_{x1x1} > 0$

$f_{x2x2} > 0$

and $f_{x1x1} * f_{x2x2} > [f_{x1x2}]^2$

y → minimum

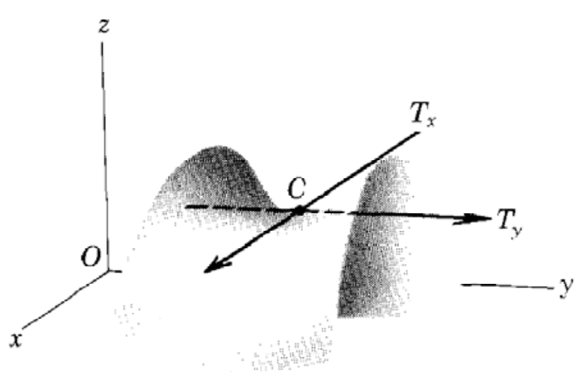
Case II:

If $f_{x1x1} < 0$

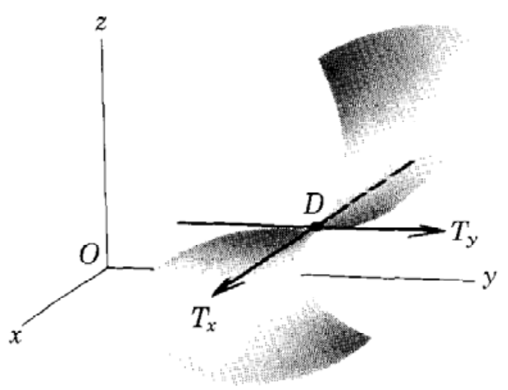
$f_{x2x2} < 0$

and $f_{x1x1} * f_{x2x2} > [f_{x1x2}]^2$

y → maximum



Saddle point



Inflection point

Case III:

If $f_{x1x1} > 0$ and $f_{x2x2} < 0$

or $f_{x1x1} < 0$ and $f_{x2x2} > 0$

So $f_{x1x1} * f_{x2x2} < 0$

$f_{x1x1} * f_{x2x2} < [f_{x1x2}]^2$

y → saddle point

Case IV:

If $f_{x1x1} < 0$ and $f_{x2x2} < 0$

or $f_{x1x1} > 0$ and $f_{x2x2} > 0$

and $f_{x1x1} * f_{x2x2} < [f_{x1x2}]^2$

y → inflection point

8.2 Condition for Maximum and Minimum

Differential:

Let $y = f(x_1, x_2)$

Find the stationary value → $dy = 0$

$$dy = f_1 dx_1 + f_2 dx_2 = 0$$

First-Order condition:

$dy = 0$ if $f_1 = f_2 = 0$

Differential of dy:

$$d(dy) \equiv d^2y = \frac{\partial(dy)}{\partial x_1} dx_1 + \frac{\partial(dy)}{\partial x_2} dx_2$$

where $\frac{\partial(dy)}{\partial x_1} = \frac{\partial}{\partial x_1}(f_1 dx_1 + f_2 dx_2) = f_{11} dx_1 + f_{21} dx_2$

$\frac{(dy)}{\partial x_2} = \frac{\partial}{\partial x_2}(f_1 dx_1 + f_2 dx_2) = f_{12} dx_1 + f_{22} dx_2$

So $d^2y = (f_{11}dx_1 + f_{21}dx_2)dx_1 + (f_{12}dx_1 + f_{22}dx_2)dx_2$

$= f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$

Note: d^2y is the second-order differential of y

dx_1^2 is the squaring of the first-order differential dx_1

$f_{12} = f_{21}$

Finding the extreme value by differential method:

First-Order condition: → find stationary value

$dy = 0 \rightarrow f_1 = f_2 = 0$

Second-Order condition:

$d^2y < 0 \rightarrow$ y is maximum

$d^2y > 0 \rightarrow$ y is minimum

From $d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$

Given that $q = d^2y$

$a = f_{11}$

$b = f_{22}$

$h = f_{12} = f_{21}$

$u = dx_1$

$v = dx_2$

$q = [u\ v] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

adding to and subtracting from the above equation by $\frac{h^2v^2}{a}$

$$q = au^2 + 2huv + \frac{h^2}{a}v^2 + bv^2 - \frac{h^2}{a}v^2$$

$$= a\left(u^2 + \frac{2h}{a}uv + \frac{h^2}{a^2}v^2\right) + \left(b - \frac{h^2}{a}\right)v^2$$

$$= a\left(u + \frac{h}{a}v\right)^2 + \frac{ab - h^2}{a}(v^2)$$

So $q > 0$ if $a > 0$ and $ab - h^2 > 0$

$q < 0$ if $a < 0$ and $ab - h^2 > 0$

$$q \text{ is } \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix} \quad \text{iff} \quad \begin{Bmatrix} a > 0 \\ a < 0 \end{Bmatrix} \quad \text{and} \quad ab - h^2 > 0$$

From q: $\begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$

where $\begin{vmatrix} a & h \\ h & b \end{vmatrix} = \begin{vmatrix} f11 & f12 \\ f21 & f22 \end{vmatrix} = f_{11}f_{22} - f_{12}^2 > 0$ → a part of 2nd order condition for both maximum and minimum

Hessian Determinant:

$$\begin{vmatrix} f11 & f12 \\ f21 & f22 \end{vmatrix} = |H|$$

Condition for Maximum and Minimum:

First-Order condition: $f_1 = f_2 = 0$

Second-Order condition:

(1) $f_{11} < 0$ and $|H| > 0$ → y is maximum
(2) $f_{11} > 0$ and $|H| > 0$ → y is minimum

More than 2 independent variables:

$$y = f(x_1, x_2, \ldots, x_n)$$

1. $f_1 = f_2 = \ldots = f_n = 0$
2. Find the solutions from 1. → stationary value
3. Find second-order derivatives of function y and form Hessian Matrix and Determinant of Principle Minor of the Hessian Matrix.

   EX: $|H1| = f_{11}$

$$|H2| = \begin{vmatrix} f11 & f12 \\ f21 & f22 \end{vmatrix}$$

$$|H3| = \begin{vmatrix} f11 & f12 & f13 \\ f21 & f22 & f23 \\ f31 & f32 & f33 \end{vmatrix}$$

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$$|Hn| = \begin{vmatrix} f11 & f12 & \ldots & f1n \\ f21 & & & \\ . & & & \\ . & & & \\ . & & & \\ fn1 & fn2 & \ldots & fnn \end{vmatrix}$$

4.

Condition	Maximum	Minimum
1^{st}-Order condition	$f_1 = f_2 = \ldots = f_n = 0$	$f_1 = f_2 = \ldots = f_n = 0$
2^{nd}-Order condition	$\|H1\| < 0$; $\|H2\| > 0$; $\|H3\| < 0$; …	$\|H1\| > 0$; $\|H2\| > 0$; $\|H3\| > 0$; …

If $d^2y < 0$ → $y = f(x_1, x_2, \ldots, x_n)$ → concave → maximum

If $d^2y > 0$ → $y = f(x_1, x_2, \ldots, x_n)$ → convex → minimum

8.3 Third Degree Price Discrimination

Assume that there are 3 markets.

Let $TR = TR_1(Q_1) + TR_2(Q_2) + TR_3(Q_3)$

$TC = TC(Q)$

where $Q = Q_1 + Q_2 + Q_3$

$¶ = TR_1(Q_1) + TR_2(Q_2) + TR_3(Q_3) - TC(Q)$

Given $¶_I = \frac{\partial ¶}{\partial Q_i}$ (I = 1, 2, 3)

$$¶_1 = TR'_1(Q_1) - TC'(Q)\frac{\partial Q}{\partial Q_1} = TR'_1(Q_1) - TC'(Q) \; ; \; \frac{\partial Q}{\partial Q_1} = 1$$

$$¶_2 = TR'_2(Q_2) - TC'(Q)\frac{\partial Q}{\partial Q_2} = TR'_2(Q_2) - TC'(Q) \; ; \; \frac{\partial Q}{\partial Q_2} = 1$$

$$¶_3 = TR'_3(Q_3) - TC'(Q)\frac{\partial Q}{\partial Q_3} = TR'_3(Q_3) - TC'(Q) \; ; \; \frac{\partial Q}{\partial Q_3} = 1$$

First-Order condition:

$$TC'(Q) = TR'_1(Q_1) = TR'_2(Q_2) = TR'_3(Q_3)$$

or $MC = MR_1 = MR_2 = MR_3$

Second-Order condition:

$$¶_{11} = TR''_1(Q_1) - TC''(Q)\frac{\partial Q}{\partial Q_1} = TR''_1(Q_1) - TC''(Q)$$

$$\P_{22} = TR''_2(Q_2) - TC''(Q)\frac{\partial Q}{\partial Q_2} = TR''_2(Q_2) - TC''(Q)$$

$$\P_{33} = TR''_3(Q_3) - TC''(Q)\frac{\partial Q}{\partial Q_3} = TR''_3(Q_3) - TC''(Q)$$

where $\frac{\partial Q}{\partial Q_i} = 1$

and $\P_{12} = \P_{21} = \P_{13} = \P_{31} = \P_{23} = \P_{32} = -TC''(Q)$

$$|H| = \begin{vmatrix} TR''1 & -TC'' & -TC'' \\ -TC'' & TR''2 - TC'' & -TC'' \\ -TC'' & -TC'' & TR''3 - TC'' \end{vmatrix}$$

1. $|H1| = TR''_1 - TC'' < 0,$

   $TR''_2 - TC'' < 0$

   and $TR''_3 - TC'' < 0.$

   (slope of MR < slope of MC)

2. $|H2| = (TR''_1 - TC'')(TR''_2 - TC'') - (TC'')^2 > 0$

   or $TR''_1TR''_2 - (TR''_1 + TR''_2)TC'' > 0$

3. $|H3| = TR''_1TR''_2TR''_3 - (TR''_1TR''_2 + TR''_1TR''_3 + TR''_2TR''_3)TC'' < 0$

Q_1, Q_2 and Q_3 → Maximize profit

8.4 The Multiple-Plant Monopolist

1 market, 2 factories

$$\P = TR(Q) - TC_1(q_1) - TC_2(q_2)$$

$= TR(q_1 + q_2) - TC_1(q_1) - TC_2(q_2)$

Max Profit:

First-Order condition:

$$\frac{\partial ¶}{\partial q_1} = TR'(q_1+q_2) - TC'_1(q_1) = o$$

$$\frac{\partial ¶}{\partial q_2} = TR'(q_1+q_2) - TC'_2(q_2) = o$$

$$TR'(q_1+q_2) = TC'_1(q_1) = TC'_2(q_2)$$

$$MR = MC_1 = MC_2$$

Second-Order condition:

$$¶_{11} = TR''(Q)\frac{\partial Q}{\partial q_1} - TC''_1(q_1) = TR''(Q) - TC''_1(q_1)$$

$$¶_{22} = TR''(Q)\frac{\partial Q}{\partial q_2} - TC''_2(q_2) = TR''(Q) - TC''_2(q_2)$$

$$¶_{12} = TR'(Q) = ¶_{21}$$

where $\frac{\partial Q}{\partial q_1} = 1$

$$|H| = \begin{vmatrix} TR''(Q)\text{-}TC''1(q1) & TR''(Q) \\ TR''(Q) & TR''(Q)\text{-}TC''2(q2) \end{vmatrix}$$

1. $|H1| = TR''(Q) - TC''_1(q_1) < 0$
2. $|H2| = [TR''(Q) - TC''_1(q_1)][TR''(Q) - TC''_2(q_2)] - [TR''(Q)]^2 > 0$

q_1 and q_2 → Maximize profit

8.5 The Multiple-Product Monopolist

A monopolist produces 2 products.

$$q_1 = f_1(p_1, p_2)$$

$$q_2 = f_2(p_1, p_2)$$

If $\frac{q_i}{\partial p_j}$ $(i \neq j) > 0 \rightarrow$ substituted goods

If $\frac{q_i}{\partial p_j}$ $(i \neq j) < 0 \rightarrow$ complementary goods

Inverse function:

$$p_1 = F_1(q_1,q_2)$$

$$p_2 = F_1(q_1,q_2)$$

$$TR_1 = p_1q_1 = TR_1(q_1, q_2)$$

$$TR_2 = p_2q_2 = TR_2(q_1, q_2)$$

$\frac{\partial TR_i}{\partial q_j}$ $(i \neq j) > 0 \rightarrow$ complementary goods

$\frac{\partial TR_i}{\partial q_j}$ $(i \neq j) < 0 \rightarrow$ substituted goods

$$\P = TR_1(q_1, q_2) + TR_2(q_1, q_2) - TC_1(q_1) - TC_2(q_2)$$

First-Order condition:

$$\frac{\partial \P}{\partial q_1} = \frac{\partial TR_1}{\partial q_1} + \frac{\partial TR_2}{\partial q_1} - TC'_1(q_1) = 0$$

$$\frac{\partial \P}{\partial q_2} = \frac{\partial TR_1}{\partial q_2} + \frac{\partial TR_2}{\partial q_2} - TC'_2(q_2) = 0$$

$$\frac{\partial TR_1}{\partial q_1} + \frac{\partial TR_2}{\partial q_1} = TC'_1(q_1)$$

$$\frac{\partial TR_1}{\partial q_2} + \frac{\partial TR_2}{\partial q_2} = TC'_2(q_2)$$

Second-Order condition:

$$\P_{11} = TR''_1 + TR''_2 - TC''_1$$

$$\P_{22} = TR''_1 + TR''_2 - TC''_2$$

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