Solution: Practice Problems for Midterm Exam

1. Determine whether the statement forms are logically equivalent. In each case, construct a truth table to justify your answer.

   (a) $(p \wedge q) \to r$, $(p \to r) \wedge (q \to r)$
   (b) $p \to (q \to r)$, $(p \to q) \to r$
   (c) $p \to q \vee r$, $p \wedge \sim q \to r$, $p \wedge \sim r \to q$.

   Answer:
   (a) $(p \wedge q) \to r$, and $(p \to r) \wedge (q \to r)$

p	q	r	$p \to r$	$q \to r$	$p \wedge q$	$(p \wedge q) \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	T	T	F	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

Since rows 4 and 6 of the truth table for $(p \wedge q) \to r$, and $(p \to r) \wedge (q \to r)$ have different truth values, then these statement forms are not logically equivalent. ■

(b)

$p \to (q \to r)$, $(p \to q) \to r$

p	q	r	$q \to r$	$p \to q$	$p \to (q \to r)$	$(p \to q) \to r$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

From the truth table, since there are cases when truth values of $p \to (q \to r)$ and $(p \to q) \to r$ (i.e. rows 6 and 8) are different, then these statements are not equivalent. ■

(c) Using the order of connective operations gives $p \to (q \vee r)$, $(p \wedge \sim q) \to r$, $(p \wedge \sim r) \to q$. Truth table:

p	q	r	$\sim q$	$\sim r$	$q \vee r$	$p \wedge \sim q$	$p \wedge \sim r$	$p \to (q \vee r)$	$(p \wedge \sim q) \to r$	$(p \wedge \sim r) \to q$
T	T	T	F	F	T	F	F	T	T	T
T	T	F	F	T	T	F	T	T	T	T
T	F	T	T	F	T	T	F	T	T	T
T	F	F	T	T	F	T	T	F	F	F
F	T	T	F	F	T	F	F	T	T	T
F	T	F	F	T	T	F	F	T	T	T
F	F	T	T	F	T	F	F	T	T	T
F	F	F	T	T	F	F	F	T	T	T

Since the last three columns of $p \to (q \vee r)$, $(p \wedge \sim q) \to r$, $(p \wedge \sim r) \to q$ have the same truth values for all possible cases in the truth table, then these statement forms are logically equivalent. ■

2. Determine whether or not the statement $\sim q \wedge p \to \sim q$ is a tautology.

Answer: The order of the connective operators has to be used here:

$$\sim q \wedge p \to \sim q \equiv ((\sim q) \wedge p) \to (\sim q).$$

Truth table:

p	q	$\sim q$	$(\sim q) \wedge p$	$((\sim q) \wedge p) \to (\sim q)$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

Notice from the table that the truth values of $\sim q \wedge p \to \sim q$ are true for all possible cases of p and q, then this statement form is a tautology. ■

3. Let p and q be statements such that $p \leftrightarrow q$ is true. Find the truth values of each of the followings statement forms and provide some justifications.
(a) $p \to q$ (b) $\sim p \to \sim q$ (c) $\sim p \wedge q$ (d) $p \vee \sim q$ (e) $\sim p \leftrightarrow q$

Answer:
Given that $p \leftrightarrow q$ is true, we know that there are two possible cases for the truth values of p and q:

- both p and q are true,
- both p and q are false.

Consider the truth values for (a)-(e) for these cases.

p	q	$\sim p$	$\sim q$	(a) $p \to q$	(b) $\sim p \to \sim q$	(c) $\sim p \wedge q$	(d) $p \vee \sim q$	(e) $\sim p \leftrightarrow q$
T	T	F	F	T	T	F	T	F
F	F	T	T	T	T	F	T	F

Notice that the truth values for each of (a)-(e) are the same for both cases. That is, we can conclude that:

(a) $p \to q$ is true (b) $\sim p \to \sim q$ is true (c) $\sim p \wedge q$ is false (d) $p \vee \sim q$ is true
(e) $\sim p \leftrightarrow q$ is false.

■

4. Consider the following statement.

If its color is green and it is edible, then it is a vegetable or it is a fruit.

(a) Write the **negation** of the above statement.

(b) Write the **contrapositive**, **inverse**, and **converse** of the above statement.

Answer:
Let p be "its color is green,"
q be "it is edible,"
r be "it is a vegetable."
s be "it is a fruit."
Then the statement can be written as

$$(p \wedge q) \to (r \vee s).$$

(a) The **negation :**

$$\sim [(p \wedge q) \to (r \vee s)] \equiv (p \wedge q) \wedge \sim (r \vee s) \equiv (p \wedge q) \wedge (\sim r \wedge \sim s)$$

"Its color is green and it is edible, but(and) it is not a vegetable and it is not a fruit."

(b) **Contrapositive:**

$$\sim (r \vee s) \to \sim (p \wedge q) \equiv (\sim r \wedge \sim s) \to (\sim p \vee \sim q)$$

"If it is not a vegetable and it is not a fruit, then its color is not green or it is not edible."

Inverse:

$$\sim (p \wedge q) \to \sim (r \vee s) \equiv (\sim p \vee \sim q) \to (\sim r \wedge \sim s)$$

"If its color is not green or it is not edible, then it is not a vegetable and it is not a fruit."

Converse

$$(r \vee s) \to (p \wedge q)$$

"If it is a vegetable or it is a fruit, then its color is green and it is edible."

5. Use truth tables to determine whether the argument forms are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer.

(a)
$$\begin{array}{ll} & p \to q \\ & q \to p \\ \therefore & p \vee q \end{array}$$

(b)
$$\begin{array}{ll} & p \\ & p \to q \\ & \sim q \vee r \\ \therefore & r \end{array}$$

(c)
If it rains, then I stay home.
If it does not rain, then I go shopping.
$\therefore$ I stay home or I go shopping.

(d)
If it rains, then I stay home.
If it does not rain, then I go shopping.
I do not go shopping.
$\therefore$ It rains.

Answer:

(a) Truth table:

		Premises		Conclusion	
p	q	$p \to q$	$q \to p$	$p \vee q$	
T	T	T	T	T	←-- critical row
T	F	F	T		
F	T	T	F		
F	F	T	T	F	←-- critical row: This has true premises but false conclusion.

Notice that the columns 3 and 4 consist of the truth values of premises and rows 1 and 4 are critical rows. Since the forth row (which is a critical row) has true premises with false conclusion, then this argument is **invalid**.

(b)
p
$p \to q$
$\sim q \vee r$
$\therefore\ r$

Truth table:

				Premises			Conclusion	
p	q	r	$\sim q$	p	$p \to q$	$\sim q \vee r$	r	
T	T	T	F	T	T	T	T	←-- critical row: This has true premises and true conclusion.
T	T	F	F	T	T	F		
T	F	T	T	T	F	T		
T	F	F	T	T	F	T		
F	T	T	F	F	T	T		
F	T	F	F	F	T	F		
F	F	T	T	F	T	T		
F	F	F	T	F	T	T		

Notice that the columns 5, 6, and 7 consist of the truth values of premises and row 1 is the only critical row. Since this critical row has true premises with true conclusion, then this argument is **valid**. ■

(c) Let p be "If it rains, then I stay home"; q be "I stay home." and r be "I go shopping." Then we can transform the given argument as follows.

	If it rains, then I stay home.		$p \to q$
	If it does not rain, then I go shopping.	$\equiv$	$\sim p \to r$ Truth table:
$\therefore$	I stay home or I go shopping.	$\therefore$	$q \vee r$

Premises Conclusion

p	q	r	$\sim p$	$p \to q$	$\sim p \to r$	$q \vee r$	
T	T	T	F	T	T	T	←-- critical row
T	T	F	F	T	T	T	←-- critical row
T	F	T	F	F	T	T	
T	F	F	F	F	T	F	
F	T	T	T	T	T	T	←-- critical row
F	T	F	T	T	F	T	
F	F	T	T	T	T	T	←-- critical row
F	F	F	T	T	F	F	

Notice that the columns 5 and 6 consist of the truth values of premises and rows 1,2,5,7 are the critical rows. Since each of these critical rows has all true premises with true conclusion, then this argument is **valid**. ■

(d) Let p be "If it rains, then I stay home"; q be "I stay home." and r be "I go shopping." Then we can transform the given argument as follows.

	If it rains, then I stay home.		$p \to q$
	If it does not rain, then I go shopping.	$\equiv$	$\sim p \to r$ Truth table:
	I do not go shopping.		$\sim r$
$\therefore$	It rains.	$\therefore$	p

Premises Conclusion

p	q	r	$\sim p$	$p \to q$	$\sim p \to r$	$\sim r$	p	
T	T	T	F	T	T	F	T	
T	T	F	F	T	T	T	T	←-- critical row
T	F	T	F	F	T	F	T	
T	F	F	F	F	T	T	T	
F	T	T	T	T	T	F	F	
F	T	F	T	T	F	T	F	
F	F	T	T	T	T	F	F	
F	F	F	T	T	F	T	F	

Notice that the columns 5, 6 and 7 consist of the truth values of premises and row 2 is the only critical row. Since this critical row has true premises with true conclusion, then this argument is **valid**. ■

6. Determine the truth value of each of these statements. Explain your answer.

(a) $\forall n \in \mathbb{Z}, \quad n-1<n$ (b) $\forall n \in \mathbb{Z}, \quad n \leq 10n$
(c) $\exists n \in \mathbb{Z}, \quad 2n=-n$ (d) $\exists n \in \mathbb{Z}^-, \quad n=\frac{1}{n}$

Answer:

(a) $\forall n \in \mathbb{Z}, \quad n-1<n$
This statement is **true**, because if we subtract both sides of inequality by n: for any $n \in \mathbb{Z}$:

$$n-1<n \qquad \Rightarrow \qquad -1<0,$$

which is true for all n. Note: other simple argument can also be used here. ■

(b) $\forall n \in \mathbb{Z}, \quad n \leq 10n$
This statement is **false**, because if let $n<0$, then $n \leq 10n$ is not true. E.g.a counterexample is $n=-1$, we have $-1<-10$, which is false. ■

(c) $\exists n \in \mathbb{Z}, \quad 2n=-n$
This statement is **true**, because we can find at least one value of $n \in \mathbb{Z}$, i.e. $n=0$, such that $2n=-n=0$. ■

(d) $\exists n \in \mathbb{Z}^-, \quad n=\frac{1}{n}$
This statement is **true**, because we can find at least one value of $n \in \mathbb{Z}^-$, i.e. $n=-1 \in \mathbb{Z}^-$, such that $n=\frac{1}{n}$ or $-1=\frac{1}{-1}$ is true. ■

7. Let $\mathbb{R}$ be the domain of x. Determine the **truth set** for each of these statements.
(a) $P(x)$: "$x+1<2x$" (b) $P(x)$: "$x^2<4$ and $x \leq 0$"

Answer:
(a) $P(x)$: "$x+1<2x$"
$P(x)$ is true when

$$x+1<2x \qquad \text{or} \qquad x+1-x<2x-x \qquad \text{or} \qquad 1<x.$$

That is, the truth set is $\{x \in \mathbb{R} | x>1\}=(1,\infty)$. ■

(b) $P(x)$: "$x^2<4$ and $x \leq 0$"
$P(x)$ is true when $x^2<4$ and $x \leq 0$ are true. Notice that

$$x^2<4 \qquad \equiv \qquad x^2-4<0 \qquad \equiv \qquad (x-2)(x+2)<0 \qquad \equiv \qquad -2<x<2.$$

That is, we want $-2<x<2$ and $x \leq 0$ to be true at the same time, which occurs when $-2<x \leq 0$ So the truth set is $\{x \in \mathbb{R} | -2<x \leq 0\}=(-2,0]$. ■

8. Let $Q(x,y)$ be the statement "$x+y=x-y$." If the domain for both variables consists of all integers, determine the truth values of the following statements. Explain you answer.

(a) $Q(1,1)$ (b) $Q(2,0)$ (c) $\forall y, Q(1,y)$

(d) $\exists x, Q(x,2)$ (e) $\forall x \exists y, Q(x,y)$ (f) $\forall y \exists x, Q(x,y)$

Answer:

(a) $Q(1,1)$:“$1+1=1-1$” or “$2=0$” is false.
(b) $Q(2,0)$:“$2+0=2-0$” or “$2=2$” is true.
(c) $\forall y, Q(1,y)$:“$1+y=1-y$” or “$y=-y$” is false (a counterexample is $y=1$).
(d) $\exists x, Q(x,2)$:“$x+2=x-2$” or “$-2=2$” is always false for any x.
(e) $\forall x \exists y, Q(x,y)$ is true because for any fixed x, we can choose $y=0$, so that “$x+0=x-0$” which is true.
(f) $\forall y \exists x, Q(x,y)$ is false because when we use a fixed value of y that is nonzero, we have “$x+y=x-y$” which will implies “$y=-y$” and this is false. A counterexample is when $y=1$, we have “$x+1=x-1$” which will implies “$1=-1$” and we cannot find any x that make this statement true.

9. Let $Q(x,y,z)$ be the statement “$x+y=z$.” Let the domain of all variables be the set of all real numbers. Determine the truth value of the statement $\exists z \forall x \forall y,\ Q(x,y,z)$. Explain your answer.

Answer:

This statement is false because there is a real number z such that“$x+y=z$” is not true. In particular, for each fixed value of z, then we can always find x and y (e.g. $x=z$, $y=1$) such that $x+y \neq z$. E.g.
$z=1$, set $x=1$, $y=1$ we will see that the statement is not true;
$z=2$, set $x=2$, $y=1$ we will see that the statement is not true;
$z=3$, set $x=3$, $y=1$ we will see that the statement is not true....etc.
Note: other counterexamples could be used too.

10. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements “x is a hummingbird,” “x is large,” “x lives on honey,” and “x is colorful,” respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ and determine whether this argument is valid or invalid by applying the equivalences of statement forms and the inference rule(s).

“All hummingbirds are colorful.”
“No large birds live on honey.”
“Birds that do not live on honey are not colorful.”
$\therefore$ “Hummingbirds are not large birds.”

Answer:

Let D be the set of all birds.
(i)“All hummingbirds are colorful.” $\equiv \forall x \in D, P(x) \rightarrow S(x)$
(ii)“No large birds live on honey.”
$\equiv \forall x \in D, Q(x) \rightarrow \sim R(x) \equiv \forall x \in D, R(x) \rightarrow \sim Q(x)$ (contrapositive)
(iii)“Birds that do not live on honey are not colorful.”
$\equiv \forall x \in D, \sim R(x) \rightarrow \sim S(x) \equiv \forall x \in D, S(x) \rightarrow R(x)$ (contrapositive)

(iv).∴ "Hummingbirds are not large birds." $\equiv \forall x \in D, P(x) \to \sim Q(x)$
So we have from (i) → (iii) → (ii), we will get the conclusion (iv): $\forall x \in D$,

$$(i)\ P(x) \to S(x) \quad , \quad (iii)\ S(x) \to R(x) \quad , \quad (ii)\ R(x) \to \sim Q(x)\ , \quad (iv)\ P(x) \to \sim Q(x),$$

and by using rule of inference **universal transitivity**, this argument is valid.

11. Let $\mathbb{Z}^+$ be the domain of x. Determine whether the following statements are true or false. Give a counterexample for each false statement.
(a) $\sqrt{x} > 1 \Rightarrow x^2 > 23$, (b) $\sqrt{x} > 2 \Leftrightarrow x^2 > 23$.

Answer:

-Notice that $\sqrt{x} > 2$ only when $x > 4$.
So " $\sqrt{x} > 2$ " is true when $x \in \{5, 6, 7, 8, \dots\}$.
Hence, truth set for predicate: " $\sqrt{x} > 2$ " is $\{5, 6, 7, 8 \dots, \}$ or $\mathbb{Z}^+/\{1, 2, 3, 4\}$.
-Notice that $x^2 > 23$ only when $x > \sqrt{23}$ for $x \in \mathbb{Z}^+$.
So "$x^2 > 23$" is true when $x \in \{5, 6, 7, 8, \dots\}$.
Hence, truth set for predicate: "$x^2 > 23$" is $\{5, 6, 7, 8 \dots, \}$ or $\mathbb{Z}^+/\{1, 2, 3, 4\}$.
(a) $\sqrt{x} > 1 \Rightarrow x^2 > 23$ is false since the truth set of "$x^2 > 23$" is not contained in the truth set of "$\sqrt{x} > 1$." In particular,

"$\sqrt{x} > 1$" is true when $x \in \{2, 3, 4, 5 \dots\}$ and hence its truth set is $T_1 := \{2, 3, 4, 5 \dots\}$.
"$x^2 > 23$" is true when $x \in \{5, 6, 7, 8 \dots, \}$ and hence its truth set is $T_2 := \{5, 6, 7, 8 \dots, \}$.

Since $T_1 \nsubseteq T_2$, this statement is false. Also, a counterexample is $x = 2$ or any integer $x \geq 2$. (Note: it is enough the just give a counterexample here).
(b) $\sqrt{x} > 2 \Leftrightarrow x^2 > 23$ is true because, from above, the truth sets of " $\sqrt{x} > 2$ " and "$x^2 > 23$" are the same.

12. Write a negation for each statement without using *the negation symbol* "∼."

(a) $\exists x \in \mathbb{R}$, $(x - 2)(x + 1) > 0$ if and only if $x > 2$ or $x < -1$.
(b) $\forall \varepsilon \exists \delta \forall x \, (|x - a| < \delta \to |f(x) - L| < \varepsilon)$

Answer:

(a) Let $P(x)$ be $(x - 2)(x + 1) > 0$, $Q(x)$ be $x > 2$, and $R(x)$ be $x < -1$. Note that

$$[P(x) \leftrightarrow (Q(x) \vee R(x))] \equiv [P(x) \to (Q(x) \vee R(x))] \wedge [(Q(x) \vee R(x)) \to P(x)]$$

$$\sim [P(x) \leftrightarrow (Q(x) \vee R(x))] \equiv \sim [P(x) \to (Q(x) \vee R(x))] \vee \sim [(Q(x) \vee R(x)) \to P(x)]$$

and since we used the fact that

$$\sim [P(x) \to (Q(x) \vee R(x))] \equiv P(x) \wedge \sim (Q(x) \vee R(x)) \equiv P(x) \wedge (\sim Q(x) \wedge \sim R(x))$$

and

$$\sim [(Q(x) \vee R(x)) \to P(x)] \equiv (Q(x) \vee R(x)) \wedge \sim P(x)$$

then the negation of the given statement is

$$\exists x \in \mathbb{R}, [P(x) \wedge (\sim Q(x) \wedge \sim R(x))] \vee [(Q(x) \vee R(x)) \wedge \sim P(x)]$$

or

$\exists x \in \mathbb{R}, [((x-2)(x+1) > 0) \wedge (x \leq 2 \wedge x \geq -1)] \vee [(x > 2 \vee x < -1) \wedge (x-2)(x+1) \leq 0]$.

(b) $\sim \forall \varepsilon \exists \delta \forall x \, (|x-a| < \delta \to |f(x) - L| < \varepsilon)$
$\equiv \exists \varepsilon \sim [\exists \delta \forall x \, (|x-a| < \delta \to |f(x) - L| < \varepsilon)]$
$\equiv \exists \varepsilon \forall \delta \sim [\forall x \, (|x-a| < \delta \to |f(x) - L| < \varepsilon)]$
$\equiv \exists \varepsilon \forall \delta \exists x \sim (|x-a| < \delta \to |f(x) - L| < \varepsilon)$
$\equiv \exists \varepsilon \forall \delta \exists x \, (|x-a| < \delta \ \wedge \ |f(x) - L| \geq \varepsilon)$

13. Show that each of the following arguments is valid by **universal modus ponens** , **universal modus tollens** or **universal transitivity**, or show that it is invalid from the **converse error** or the **inverse error**. In addition, use also the **diagram** to confirm that each argument is valid or invalid.

(a) All rabbits like vegetable. My pet is not a rabbit. Therefore, my pet does not like vegetable.
(b) Everyone who eats fruit every day is healthy. Linda is not healthy. Therefore, Linda does not eat fruit every day.

Answer:

(a) We can transform the given argument in the quantified form of **inverse error:**

$$\forall x, P(x) \to Q(x)$$
$$\sim P(a) \text{ for a particular } a = \text{ my pet}$$
$$\therefore \sim Q(a)$$

where $P(x)$ is defined as "x is a rabbits" and $Q(x)$ is defined as "x likes vegetable." Hence the argument is **invalid**.

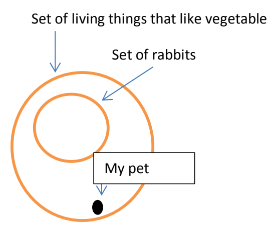

Figure 1: Problem 8 (a)

From the diagram, since it is possible that the "my pet " is still inside the set of living things that like vegetable, even if "my pet " is not in the set of rabbits, then the statement is **invalid**.

(b) First the given premise " Everyone who eats fruit every day is healthy" can be written in terms of *if-then statement* as "$\forall x$, if x eats fruits everyday, then x is healthy."

Then, the given argument can be re-written in a valid form of **universal modus tollens**:

$$\forall x, P(x) \to Q(x)$$
$$\sim Q(a) \text{ for a particular } a = \text{Linda}$$
$$\therefore \sim P(a),$$

where $P(x)$ is defined as "x eats fruit everyday" and $Q(x)$ is defined as "x is healthy." Hence the argument is **valid** by **universal modus tollens**.

Diagram

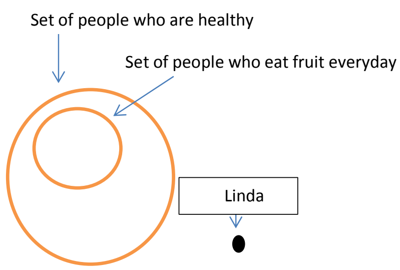

Figure 2: Problem 8 (b)

From the diagram, since Linda is not in the set of people who are healthy, which contains the set of people who eats fruit everyday. So it is impossible that Linda is in the set of people who eats fruit everyday. Hence the conclusion that Linda does not eat fruit everyday is **valid**.

14. (a) Prove the statement:
"There is a pair of real numbers x and y such that $\lfloor x-y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. "

(b) Disprove the statement: "For all real numbers x and y, $\lfloor x-y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. "

Answer:

(a) To prove this *existential statement,* we can find a pair of x and y such that the given statement is true.
Let $x = 0$ and $y = 0$. Then $\lfloor x-y \rfloor = \lfloor 0-0 \rfloor = \lfloor 0 \rfloor = 0$, and $\lfloor x \rfloor = \lfloor 0 \rfloor = 0$ and $\lfloor y \rfloor = \lfloor 0 \rfloor = 0$. That is,

$$\lfloor 0-0 \rfloor = \lfloor 0 \rfloor - \lfloor 0 \rfloor$$

and hence the statement is true. ■

(b) To disprove the *universal statement*, we can find a pair of x and y such that the given statement is false.
Consider

$$\lfloor 1 \rfloor = 1, \quad \lfloor 0.5 \rfloor = 0, \text{ which imply } \lfloor 1 \rfloor - \lfloor 0 \rfloor = 1$$

but

$$\lfloor 1 - 0.5 \rfloor = \lfloor 0.5 \rfloor = 0.$$

That is, a counterexample is $x = 1, y = 0.5$. ■

15. Show that "for any integer n, if $n^3 + 5$ is odd, then n is even," by using
a) a proof by contraposition,
b) a proof by contradiction.

Answer

a) **Proof by contraposition**: To prove by a contraposition, we consider the contrapositive of the given statement:

for any integer n, if n is odd, then $n^3 + 5$ is even.

We can do this by direct proof. Suppose n is odd. Then, we can write

$$n = 2k + 1,$$

where k is an integer and

$$n^3 + 5 = (2k+1)^3 + 5 = (2k)^3 + 3(2k)^2 + 3(2k) + 1 + 5 = 2(4k^3 + 6k^2 + 3k + 3).$$

That is, we can write $n^3 + 5$ in terms of $n^3 + 5 = 2s$ where $s = 4k^3 + 6k^2 + 3k + 3$ is an integer (since it is the product and the sum of integers). Hence, $n^3 + 5$ is an even integer. ■

b) **Proof by contradiction**:

Suppose not. I.e., suppose the negation " $n^3 + 5$ is odd, but n is not even" is true. Then, n is odd and we can write

$$n = 2k + 1$$

for some integer $k \in \mathbb{Z}$. So we have

$$n^3 + 5 = (2k+1)^3 + 5 = (2k)^3 + 3(2k)^2 + 3(2k) + 1 + 5 = 2(4k^3 + 6k^2 + 3k + 3),$$

which implies that $n^3 + 5$ is an even integer (since we can write $n^3 + 5$ in terms of $n^3 + 5 = 2s$ where $s = 4k^3 + 6k^2 + 3k + 3$ is an integer).
This **contradicts** to the fact that $n^3 + 5$ is odd. Hence, the negation is false and cannot happen. That is, the given statement is true. ■

16. Prove by the **method of exhaustion** that " $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$."

Answer Given that an integer n such that $1 \leq n \leq 4$ implies that $n = 1, 2, 3,$ or 4.

For $n = 1$, $1^2 + 1 = 2^1$ and $n^2 + 1 \geq 2^n$ is true.

For $n = 2$, $2^2 + 1 = 5$ and $2^2 = 4$, so $n^2 + 1 \geq 2^n$ is true.

For $n = 3$, $3^2 + 1 = 10$ and $2^3 = 8$, so $n^2 + 1 \geq 2^n$ is true.

For $n = 4$, $4^2 + 1 = 17$ and $2^4 = 16$, so $n^2 + 1 \geq 2^n$ is true.

Therefore, $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$. ■

17. Use the **proof by cases** to show that " for any integer n, $n^2 \geq n$."
[Hint: Consider 3 cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$]
Answer We will prove by cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$.

Case (i): $n \in \mathbb{Z}^-$
In this case, $n < 0$ and $n^2 > 0$. That is $n < 0 < n^2$ and $n^2 \geq n$ is true.

Case (ii): $n = 0$
In this case, $n^2 = 0$. So $n = n^2$ and $n^2 \geq n$ is true.

Case (iii): $n \in \mathbb{Z}^+$
In this case, consider $n^2 - n = n(n-1)$. Since $n \in \mathbb{Z}^+$ implies that n is a positive integer and the smallest value of n is 1. That is $n \geq 1$, which implies $n - 1 \geq 0$. Since $n > 0$ and $n - 1 \geq 0$,

$$n^2 - n = n(n-1) \geq 0, \text{ which implies} n^2 \geq n,$$

and the given statement is true.

Since any integer n can be in either case (i), (ii), or (iii), then the given statement is true. ■

18. Consider the statement: for $n \geq 1$,

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

Suppose we want to prove the above statement by **mathematical induction**.

(a) What is $P(n)$?
(b) Write $P(1)$: Is $P(1)$ true?
(c) Write $P(k)$:
(d) Write $P(k+1)$:
(e) Prove the above statement: $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$, by using **mathematical induction**.

Answer:

(a) $P(n)$ is a statement

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n},$$

for $n \geq 1$

(b) Write $P(1)$:

$$\frac{1}{2} = \frac{2^(1-1)}{2^1},$$

Yes, $P(1)$ is true because $\frac{2^(1-1)}{2^1} = \frac{1}{2}$.

(c) $P(k)$:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}.$$

(d) $P(k+1)$:

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{k+1}}=\frac{2^{k+1}-1}{2^{k+1}}.$$

(e) Prove the above statement: $\sum_{j=1}^{n}\frac{1}{2^j}=\frac{2^n-1}{2^n}$, by using **mathematical induction**.

Let $P(n)$ be the statement $\sum_{j=1}^{n}\frac{1}{2^j}=\frac{2^n-1}{2^n}$.

(I) Basis step: Show that $P(1)$ is true. $P(1)$:

$$\frac{1}{2}=\frac{2^(1-1)}{2^1},$$

$P(1)$ is true because $\frac{2^(1-1)}{2^1}=\frac{1}{2}$.

(II) Inductive step: Show that if $P(k)$ is true, then $P(k+1)$ is true.
Assume that $P(k)$ is true, or

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^k}=\frac{2^k-1}{2^k}.$$

We want to show that $P(k+1)$:

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{k+1}}=\frac{2^{k+1}-1}{2^{k+1}}.$$

$$\begin{aligned}\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{k+1}} &= \underbrace{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^k}}_{=\frac{2^k-1}{2^k}}+\cdots+\frac{1}{2^{k+1}} \quad \text{by inductive hypothesis } P(k)\\ &= \frac{2^k-1}{2^k}+\frac{1}{2^{k+1}}\\ &= \frac{2(2^k-1)+1}{2^{k+1}}\\ &= \frac{(2^{k+1}-2)+1}{2^{k+1}}\\ &= \frac{2^{k+1}-1}{2^{k+1}}\end{aligned}$$

and the statement $P(k+1)$ is true.
From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 1$ by mathematical induction proof. ■

19. Use mathematical induction proof to show that

$$n! < n^n,$$

for any integer n that is greater than 1.

Answer:
Proof by mathematical induction:

Let $P(n)$ be the statement $n! < n^n$. We want to prove that $P(n)$ is true for all integer $n > 1$. Note that for $n \in \mathbb{Z}$, $n > 1$ is equivalent to $n \geq 2$ and we therefore have to use $n = 2$ in the basis step.

(I) **Basis step:** Show that $P(2)$ is true.
$P(2)$: $2! < 2^2$.
Since $2! = 2$ and $2^2 = 4$. Hence $2! < 2^2$ and $P(2)$ is true.

(II)**Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 2$.

Assume that $P(k) : k! < k^k$ is true. ————————(★) "inductive hypothesis"
We want to show that $P(k+1) : (k+1)! < (k+1)^{(k+1)}$ is true. Consider

$$
\begin{aligned}
(k+1)! &= k!(k+1) \\
&< k^k(k+1) && \text{by } (\bigstar) \text{ "inductive hypothesis"} \\
&< (k+1)^k \cdot (k+1) && k^k < (k+1)^k \text{ for } k \geq 2 \\
&= (k+1)^{(k+1)}
\end{aligned}
$$

Note: we have used the fact that since $k < k+1$ implies $k^k < (k+1)^k$ (using the same exponent).
Therefore $P(k+1)$ is true.
Note also that we have used $k!(k+1) = \underbrace{1 \cdot 2 \cdot 3 \cdot \dots \cdot k}_{k!} \cdot (k+1) = (k+1)!$.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 2$ by mathematical induction proof. ■