Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} \; E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$\max_{C_s, W_s, \forall t} E_t\left[\sum_{s=t}^{T-1} \delta^s \left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + \delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right] ; \quad U(C_t, t) = \delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)$$

$$U_c(C_t, t) = \delta^s C_t^{-\gamma}$$

FOC

Date T-1, $U_c(C_{T-1}, T-1) = E_{T-1}[B_W(W_T, T) R_{T-1}]$

$$\delta^{T-1}\left(C_{T-1}^{-\gamma}\right) = E_{T-1}\left[\delta^T (R_{T-1}) \cdot W_{T-1}^{-\gamma}\right]$$

$$= E_{T-1}\left[\delta^T (R_{T-1}) \cdot (S_{T-1} \cdot R_{T-1})^{-\gamma}\right]$$

$$\delta^{T-1} \cdot C_{T-1}^{-\gamma} = \delta^T \cdot S_{T-1}^{-\gamma} \cdot R_{T-1}^{1-\gamma}$$

$$C^*_{T-1} = \delta \cdot W_{T-1}^{-\gamma} \; ※$$

Optimality Conditions for $\{\omega^*_{i,T-1}\}$

$$E_{T-1}\left[\frac{R_{i,T-1}}{R^*_{T-1}}\right] = R_{f,T-1} E_{T-1}[\cdot]$$

$$\delta^{T-1}(C_{T-1}^{-\gamma}) = R_{f,T-1} \delta^T E_{T-1}\left[R^*_{T-1} W_{T-1}^{-\gamma}\right]$$

$$1 = \frac{\delta \cdot W_{T-1}^{-\gamma}}{C_{T-1}^{-\gamma}} E_{T-1}\left[R^*_{T-1}\right] \; ※$$

$$W_{t+1} = W_t + y_t - C_t\left(R_{ft} + \sum_{i=1}^{n} \omega_{it} \underbrace{(R_{it} - R_{ft})}_{\text{premium}}\right)$$

$$= S_t R_t$$

Score.....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

Bellman equation

$$J(W_t, t) = \max_{C_t, \{w_i, t\}} U(C_t, t) + E_t\left[J(W_{t+1}, t+1)\right]$$

$$J(W_{T-1}, T-1) = \max_{C_{T-1}, \{w_i, T-1\}} E_{T-1}\left[U(C_{T-1}, T-1) + B(W_T, T)\right]$$

$$= \max_{C_{T-1}, \{w_i, T-1\}} U(C_{T-1}, T-1) + E_{T-1}\left[B(W_T, T)\right]$$

$$J(W_{T-1}, T-1) = \max_{C_{T-1}, \{w_i, T-1\}} U(C_{T-1}, T-1) + E_{T-1}\left[B(S_{T-1} R_{T-1}, T)\right]$$

note: $S_{T-1} = W_{T-1} + y_{T-1} - C_{T-1}$

$R_{T-1} = R_{f,T-1} + \sum_{i=1}^{n} w_{i,T-1}(R_{i,T-1} - R_{f,T-1})$

$$= \delta^{s}\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + E_{T-1}\left[\delta^{T}\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

$$= \delta^{T-1}\left(\frac{C_{T-1}^{(1-\gamma)}}{1-\gamma}\right) + \delta^{T} E_{T-1}\left[\frac{\left[R^*_{T-1}(W_{T-1} - C^*_{T-1})\right]^{(1-\gamma)}}{1-\gamma}\right]$$

$$= \delta^{T-1}\left(\frac{\delta^{1-\gamma}}{1-\gamma}\right) + \delta^{T}\left(E_{T-1}\left[R_{T-1}^{(1-\gamma)^*}\right] + W_{T-1}^{(1-\gamma)} - \delta\right)$$

$$= \frac{\delta^{T-\gamma}}{1-\gamma} + \delta^{T}\left(E_{T-1}\left[R_{T-1}^{(1-\gamma)^*}\right] + W_{T-1}^{(1-\gamma)} - \delta\right) \#$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$\max_{C_s, w_s, \forall t} E_t\left[\sum_{s=t}^{T-1} \delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + \delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right] \;; \quad U(C_t, t) = \delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)$$

$$U_c(C_t, t) = \delta^s C_t^{-\gamma}$$

FOC

Date T-2, $U_c(C_{T-2}, T-2) = E_{T-2}[B_W(W_T, T) R_{T-2}]$

$$\delta^{T-2}\left(C_{T-2}^{-\gamma}\right) = E_{T-2}\left[\delta^T (R_{T-2}) \cdot W_{T-2}^{-\gamma}\right]$$

$$= E_{T-2}\left[\delta^T (R_{T-2}) \cdot (S_{T-2} \cdot R_{T-2})^{-\gamma}\right]$$

$$\delta^{T-2} \cdot C_{T-2}^{-\gamma} = \delta^T \cdot S_{T-2}^{-\gamma} \cdot R_{T-2}^{1-\gamma}$$

$$C^*_{T-2} = \delta \cdot W_{T-2}^{-\gamma} \divideontimes$$

Optimality Conditions for $\{w^*_{i,T-}\}$

$$E_{T-2}\left[\frac{R_{i,T-2}}{R^*_{T-2}}\right] = R_{f,T-2} E_{T-2}[\,\cdot\,]$$

$$\delta^{T-2}\left(C_{T-2}^{-\gamma}\right) = R_{f,T-2}\, \delta^T E_{T-2}\left[R_{T-2} W_{T-2}^{-\gamma}\right]$$

$$1 = \frac{\delta \cdot W_{T-2}^{-\gamma}}{C_{T-2}^{-\gamma}} E_{T-2}[R_{T-2}] \divideontimes$$

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2},T-2) = \max_{C_{T-2},\{\omega_i,T-2\}} \left\{U(C_{T-2},T-2)+E_{T-2}\left[\max_{C_{T-2},\{\omega_i,T-2\}} E_{T-1}[U(C_{T-1},T-1)+B(W_T,T)]\right]\right\}$$

$$J(W_{T-2},T-2) = \max_{C_{T-2},\{\omega_i,T-2\}} \left\{U(C_{T-2},T-2)+E_{T-2}\left[J(W_{T-1},T-1)\right]\right\}$$

$$U_C(C^*_{T-2},T-2) = E_{T-2}[J_W(W_{T-1},T-1)R_{T-2}]$$
$$= R_{f,T-2}E_{T-2}[J_W(W_{T-1},T-1)]$$
$$= J_W(W_{T-2},T-2)$$

$\therefore$ marginal utility of current consumption U_C = marginal utility of wealth (future consumption)

FOC: $E_{T-2}[R_{i,T-2}\,J(W_{T-1},T-1)]$

$= R_{f,T-2}\,E_{T-2}[J(W_{T-1},T-1)]$

$$J(W_{T-2},T-2) = \delta^{s}\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+E_{T-1}\left[\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

$$= \delta^{T-2}\left(\frac{C_{T-2}^{(1-\gamma)}}{1-\gamma}\right)+\delta^{T-2}\left(E_{T-2}[R_{T-2}^{(1-\gamma)*}]+\delta+W_{T-2}^{1-\gamma}\right)^*$$

So, Optimal conditions for any date T-t ; t = 1,2,...

$$C_t^* = \delta W_t^{-\gamma}$$

$$1 = \frac{\delta W_t^{-\gamma}}{C_t^{-\gamma}}E_t[R_t^*]$$