

EE320 Lecture Note
Chapter 8: Optimization without Constraint:
More-Than-One-Independent-Variable Case

Recall from one-independent variable optimization without constraints:

$$\begin{aligned}y &= f(x) \\ dy &= f'(x)dx\end{aligned}$$

FOC for an optimum necessary condition

$$\frac{dy}{dx} = f'(x) = 0 \text{ or first-order differential condition for any } dx \neq 0; dy = 0$$

SOC to check sufficient condition

$$\frac{d^2y}{dx^2} = f''(x) < 0 \text{ maximum}$$

$$f''(x) > 0 \text{ minimum}$$

$$f''(x) < 0 \text{ indetermined (max,min,inflection point)}$$

$$\rightarrow d^2y = d(dy) = d(f'(x)dx) = (df'(x))dx = f''(x)dx$$

1 Two-choice-variable optimization

$$z = f(x,y)$$

First-order-necessary condition is $dz = f_x dx + f_y dy = 0$
 $\therefore dz = 0$ when $f_x = f_y = 0$

Theorem: A differentiable function $z = f(x, y)$ can only have a maximum or minimum at an interior point (x_0, y_0) if it is a stationary point. That is, if the point $(x, y) = (x^*, y^*)$ satisfies the two FOC equations: $f_x(x_0^*, y_0^*) = 0$

$$f_y(x_0^*, y_0^*) = 0$$

ex. $z = f(x, y) = 10x + 10y + xy - x^2 - y^2$

$$FOC : \frac{\partial z}{\partial x} = 10 + y - 2x = 0 \rightarrow \textcircled{1} \Rightarrow y = 2x - 10$$

$$\frac{\partial z}{\partial y} = 10 + x - 2y = 0 \rightarrow \textcircled{2}$$

Stationary points (or critical points) are : substitute $\textcircled{1}$ into $\textcircled{2}$

$$\begin{aligned} 10 + x - 2(2x - 10) &= 0 \\ x^* &= 10 \\ y^* &= 10 \end{aligned}$$

Next, find whether those points give max,min or saddle point.

$\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ are necessary but not yet sufficient for max/min.

2 Second-Order Conditions

$$\begin{aligned} d^2z &= d(dz) \quad \text{suppose } z = f(x, y) \\ &= \frac{\partial}{\partial x} dz dx + \frac{\partial}{\partial y} dz dy \\ &= \frac{\partial}{\partial x} (f_x dx + f_y dy) dx + \frac{\partial}{\partial y} (f_x dx + f_y dy) dy \\ &= (f_{xx} dx + f_{xy} dy) dx + (f_{yx} dx + f_{yy} dy) dy \\ &= f_{xx} dx^2 + f_{xy} dy dx + f_{yx} dx dy + f_{yy} dy^2 \\ d^2z &= f_{xx} dx^2 + 2f_{xy} dy dx + f_{yy} dy^2 \\ &= \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\ &\quad \downarrow \\ &\text{Hessian matrix or principal minor} \end{aligned}$$

First leading principal minor $|H_1| = f_{xx}$

Second leading principal minor $|H_2| = f_{xx}f_{yy} - f_{xy}^2$

$$\begin{aligned} \text{Note } q &= au^2 + 2huv + bv^2 \\ &= au^2 + 2huv + \frac{h^2}{a}v^2 + bv^2 - \frac{h^2}{a}v^2 \\ &= a\left(u^2 + \frac{2h}{a}uv + \frac{h^2}{a^2}v^2\right) - \left(b - \frac{h^2}{a}\right)v^2 \\ &= a\left(u + \frac{h}{a}v\right)^2 + \left(\frac{ab-h^2}{a}\right)v^2 \end{aligned}$$

$q > 0$ when $a > 0$ and $ab - h^2 > 0$

$q < 0$ when $a < 0$ and $ab - h^2 > 0$

$$\begin{aligned} q &= d^2z, & u &= dx \\ a &= f_{xx}, & y &= dy \\ b &= f_{yy}, & h &= f_{xy} \end{aligned}$$

Second-order sufficient condition

Given that FOC is satisfied

① maximum when $d^2z < 0$ iff $f_{xx} < 0$

$$f_{yy} < 0$$

$$f_{xx}f_{yy} > (f_{xy})^2 \text{ or } |H_2| > 0$$

② minimum when $d^2z > 0$ iff $f_{xx} > 0$

$$f_{yy} > 0$$

$$f_{xx}f_{yy} > (f_{xy})^2 \text{ or } |H_2| > 0$$

③ otherwise, saddle point

ex. $f(x, y) = 10x + 10y + xy - x^2 - y^2$

$$FOC : \left. \begin{array}{l} f_x = 10 + y - 2x \\ f_y = 10 + x - 2y \end{array} \right\} x^* = 10, y^* = 10$$

$$SOC : \left. \begin{array}{l} f_{xx} = -2, f_{yy} = -2 \\ f_{xy} = 1, f_{yx} = 1 \end{array} \right\} H = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$|H_1| = -2$$

$$|H_2| = 4 - 1 = 3 > 0$$

$$\text{Since } f_{xx} < 0, (-2 < 0) \Rightarrow d^2z < 0$$

$$f_{xy} < 0, (-2 < 0)$$

$$f_{xx}f_{yy} > (f_{xy})^2, (4 > 1)$$

$$\underline{\text{ex.}} \quad f(x, y) = -2x^2 - 2xy - 2y^2 + 32x + 42y - 158$$

$$x^* = 5$$

$$y^* = 8$$

$$\left. \begin{array}{l} |H_1| < 0 \\ |H_2| = 12 > 0 \end{array} \right\} \max$$

$$\underline{\text{ex.}} \quad f(x, y) = -x^2 + xy - y^2 + 3x$$

$$(2, 1) \quad z^* = 3$$

$$|H_1| < 0$$

$$|H_2| > 0$$

3 Concavity and Convexity: 2-independent-variable-case

Consider $f(x_1, x_2)$

Definition the function is (strictly) concave iff for any pair of distinct points M & N on its graph or surface, line segment MN lies either below or entirely below

Definition the function is (strictly) convex iff for any pair of distinct points M & N on its graph or surface, line segment MN lies either above or entirely above

$$\text{or } \overbrace{\alpha f(x) + (1 - \alpha)f(x')}^{\text{"Height of MN"}} < \overbrace{f(\alpha x + (1 - \alpha)x')}^{\text{"Height of arc"}} \Rightarrow \text{strictly concave}$$

$$\alpha f(x) + (1 - \alpha)f(x') > f(\alpha x + (1 - \alpha)x') \Rightarrow \text{strictly convex}$$

If we use derivative conditions;

Consider

$z = f(x_1, x_2)$ - twice continuously differentiable $\rightarrow d^2z$ is defined.

$z = f(x_1, x_2)$ is strictly concave iff d^2z is everywhere definite negative or $d^2z < 0$

$\Rightarrow z^*$ is max.

$z = f(x_1, x_2)$ is strictly convex iff d^2z is everywhere definite positive or $d^2z > 0$

$\Rightarrow z^*$ is min.

ex. $z = x_1^2 - x_2^2$

$$\left. \begin{array}{l} f_{11} = 2 \\ f_{12} = f_{21} = 0 \\ f_{22} = 2 \end{array} \right\} H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad \left. \begin{array}{l} f_{xx} > 0 \\ f_{yy} < 0 \\ f_{xx}f_{yy} - f_{xy}^2 < 0 \end{array} \right\} \therefore \text{neither concave nor convex}$$

ex. $z = x_1^2 - x_2^2$

$$\left. \begin{array}{l} f_{11} = f_{22} = 2 \\ f_{12} = f_{21} = 0 \end{array} \right\} H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \left. \begin{array}{l} f_{xx} > 0 \\ f_{yy} > 0 \\ f_{xx}f_{yy} - f_{xy}^2 > 0 \end{array} \right\} \therefore d^2z > 0 \Rightarrow \text{strictly convex}$$

ex. $z = 2x^2 - xy + y^2$ (convex)

(Application: Duopoly)

There are 2 firms with identical cost $TC_i = cQ_i$, $i = 1, 2$ and

market demand $P = a - bQ$, $Q = Q_1 + Q_2$

Find Q_1, Q_2 that maximizes each firm's profit.

$$\begin{aligned} \Rightarrow \text{Firm 1: } \Pi_1 &= PQ_1 - cQ_1 \\ &= [a - b(Q_1 + Q_2)]Q_1 - cQ_1 \end{aligned}$$

$$\Pi_1 = (a - c)Q_1 - bQ_1^2 - bQ_1Q_2$$

$$\max_{Q_1} \Pi_1 = (a - c)Q_1 - bQ_1^2 - bQ_1Q_2$$

$$\underline{FOC} \quad \frac{\partial \Pi}{\partial Q_1} = (a - c) - 2bQ_1 - bQ_2 = 0$$

$$Q_1^* = \frac{(a-c)-bQ_2}{2b} \quad \text{firm's best response function}$$

$$\Rightarrow \text{Firm 2: } \Pi_2 = PQ_2 - cQ_2$$

$$\underline{FOC} \quad Q_2^* = \frac{(a-c)-bQ_1}{2b} \quad \text{firm's best response function}$$

$$\textcircled{1} : Q_1 + \frac{Q_2}{2} = \frac{a-c}{2b}$$

$$\textcircled{2} : \frac{Q_1}{2} + Q_2 = \frac{a-c}{2b}$$

$$\Rightarrow \begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \frac{a-c}{2b} \\ \frac{a-c}{2b} \end{bmatrix}$$

$$\Rightarrow Q_1^* = \frac{a-c}{3b} \quad , \quad Q_2^* = \frac{a-c}{3b}$$

ex. market demand $P = 150 - Q$, $Q = Q_1 + Q_2$, $P = P_1 = P_2$

Check for SOC : $TC_1 = 60Q_1$, $TC_2 = 60Q_2$, Q_1^*, Q_2^* ?

$$\Rightarrow Q_1^* = \frac{90-Q_2}{2}, \quad Q_2^* = \frac{90-Q_1}{2}$$

$$\left. \begin{array}{l} Q_1^* = 30 \\ Q_2^* = 30 \end{array} \right\} Q = 60$$

$\Rightarrow d\Pi < 0$ max. ?

(Application: Duopoly-extension)

1. Cournot: move at the same time
2. Stackelberg: one seller can move first

ex.

$P = 30 - Q$ two firms

$$Q = q_1 + q_2$$

$$MC_1 = MC_2 = 3$$

i) move simultaneously

$$TR_1 = P \cdot q_1 = (30 - Q)q_1 = (30 - q_1 - q_2)q_1 = 30q_1 - q_1^2 - q_1q_2$$

$$TC_1 = 3q_1 + 5$$

$$\Pi = TR - TC = 30q_1 - q_1^2 - q_1q_2 - 3q_1 - 5$$

$$\left. \begin{array}{l} \frac{\partial \Pi}{\partial q_1} = 30 - 2q_1 - q_2 - 3 = 0 \Rightarrow q_1 = 13.5 - 0.5q_2 \\ \text{Repeat the same process for } q_2 \Rightarrow q_2 = 13.5 - 0.5q_1 \end{array} \right\} \text{reaction functions}$$

Solve for $q_1^* = 13.5 - 0.5(13.5 - 0.5q_1) \Rightarrow q_1^* = 9$, $q_2^* = 9 \Rightarrow Q = 18, P = 12$

$$\Pi_1^* = 30 \cdot 9 - 81 - 81 - 27 - 5 = 76$$

$$\Pi_2^* = 30 \cdot 9 - 81 - 81 - 27 - 5 = 76$$

ii) firm 1 moves first:

$$\begin{aligned} TR_1 &= 30q_1 - q_1^2 - q_1q_2 \\ q_2 &= 13.5 - 0.5q_1 \end{aligned}$$

perfect knowledge \Rightarrow 1 knows reaction function of 2. So take 2's behaviour into account.

$$\left. \begin{aligned} TR_1 &= 30q_1 - q_1^2 - q_1(13.5 - 0.5q_1) = 16.5q_1 - 0.5q_1^2 \Rightarrow MR = 16.5 - q_1 \\ TC &= 3q_1 + c \qquad \qquad \qquad \Rightarrow MC = 3 \end{aligned} \right\} MR = MC$$

$$\Rightarrow q_1 = 13.5$$

$$\text{Firm 2 will realize and react } q_2 = 13.5 - 0.5(13.5) = 6.75$$

$$Q = 13.5 + 6.75 = 20.25$$

$$P = 30 - 20.25 = 9.75$$

$$\Pi_1 = 16.5(13.5) - 0.5(13.5)^2 - 3(13.5) - 5 = 86.125$$

$$\Pi_2 = 9.75(6.75) - 3(6.75) - 5 = 58.5625$$

iii) firm 2 moves first:

$$\begin{aligned} \text{Similarly, } q_2 &= 13.5 \\ q_1 &= 6.75 \\ Q &= 20.25 \\ P &= 9.75 \\ \Pi_2 &= 86.125 \\ \Pi_1 &= 58.5625 \end{aligned}$$

iv) both firms try to lead

$$\begin{aligned} q_1 &= 13.5 \\ q_2 &= 13.5 \\ Q &= 27 \\ P &= 3 \\ \Pi_1 &= 3(13.5) - 3(13.5) - 5 = -5 \\ \Pi_2 &= -5 \end{aligned}$$

(Application: Multiproduct firm)

PC $P_1 = 6$, $P_2 = 9$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$. Find Q_1 and Q_2 that max Π

$$\Pi(Q_1, Q_2) = TR_1 + TR_2 - TC$$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2) = 6Q_1 + 9Q_2 - (2Q_1^2 + Q_1Q_2 + 2Q_2^2)$$

$$\text{FOC} : \left. \begin{array}{l} \frac{\partial \Pi}{\partial Q_1} = 6 - 4Q_1 - Q_2 = 0 \\ \frac{\partial \Pi}{\partial Q_2} = 9 - Q_1 - Q_2 = 0 \end{array} \right\} \begin{array}{l} 4Q_1 + Q_2 = 6 \\ Q_1 + 4Q_2 = 9 \end{array}$$

$$\Rightarrow \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} \Rightarrow Q_1^* = 1, \quad Q_2^* = 2$$

$$\text{SOC} : \left. \begin{array}{l} \Pi_{11} = -4 \\ \Pi_{22} = -4 \\ \Pi_{12} = -1 \end{array} \right\} \begin{array}{l} \Pi_{11} < 0 \\ \Pi_{22} < 0 \\ \Pi_{11}\Pi_{22} \cdot \Pi_{12}^2 > 0 \end{array} \Rightarrow d^2\Pi < 0 \text{ max}$$

Monopoly : $P_1 = 35 - Q_1$, $P_2 = 33 - Q_2$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.
Find Q_1 and Q_2 that max Π

$$\Pi(Q_1, Q_2) = TR_1 + TR_2 - TC$$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2) = (35 - Q_1)Q_1 - (33 - Q_2)Q_2 - (2Q_1^2 + Q_1Q_2 + 2Q_2^2)$$

$$\text{FOC} : \left. \begin{array}{l} \frac{\partial \Pi}{\partial Q_1} = 35 - 2Q_1 - Q_2 = 0 \\ \frac{\partial \Pi}{\partial Q_2} = 33 - 2Q_2 - Q_1 = 0 \end{array} \right\} \begin{array}{l} 6Q_1 + Q_2 = 35 \\ Q_1 + 2Q_2 = 33 \end{array}$$

$$\Rightarrow \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \end{bmatrix} \Rightarrow Q_1^* = \frac{177}{35} = 5.05, \quad Q_2^* = 4.66$$

$$\text{SOC} : \left. \begin{array}{l} \Pi_{11} = -6 < 0 \\ \Pi_{22} = -6 < 0 \\ \Pi_{12} = -1 \end{array} \right\} \Rightarrow \Pi_{11}\Pi_{22} \cdot \Pi_{12}^2 > 0 \Rightarrow d^2\Pi < 0 \text{ max}$$

(Application: Multiplant firm)

Let $P = 25$, $TC_1 = 2Q_1^2 + 5Q_1 + 10$, $TC_2 = 2Q_2^2 + 3Q_2 + 15$

Find Q_1 and Q_2 that maximize firm's profit

$$\Pi(Q_1, Q_2) = TR_1 + TC_1 - TC_2$$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2) = 25(Q_1 + Q_2) - (2Q_1^2 + 5Q_1 + 10) - (2Q_2^2 + 3Q_2 + 15)$$

$$\text{FOC} : \frac{\partial \Pi}{\partial Q_1} = 20 - 4Q_1 = 0 \Rightarrow Q_1^* = 5$$

$$\frac{\partial \Pi}{\partial Q_2} = 22 - 4Q_2 = 0 \Rightarrow Q_2^* = 5.5$$

$$\left. \begin{array}{l} \text{SOC} : \Pi_{11} = -4 < 0 \\ \Pi_{22} = -4 < 0 \\ \Pi_{12} = 0 \end{array} \right\} H = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}, |H| = 16 > 0 \quad \therefore d^2\Pi < 0 \quad \max$$

(Application: Multimarket Monopoly or Price discrimination)

Let $R = R_1(C_1) + R_2(C_2)$ and $C = C(Q)$ where $Q = Q_1 + Q_2$.

Find FONC and SOSC for Π max.

$$\Pi = R_1(Q_1) - R_2(Q_2) - C(Q)$$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2)$$

FONC

$$\left. \begin{array}{l} \frac{\partial \Pi}{\partial Q_1} = R_1'(Q_1) - C'(Q) \frac{\partial Q}{\partial Q_1} = 0 \Rightarrow R_1'(Q_1^*) = C'(Q^*) \\ \frac{\partial \Pi}{\partial Q_2} = R_2'(Q_2) - C'(Q) \frac{\partial Q}{\partial Q_2} = 0 \Rightarrow R_2'(Q_2^*) = C'(Q^*) \end{array} \right\}$$

Solve for (Q_1^*, Q_2^*) makes $MR_1 = MR_2 = MC$

SOSC

$$\Pi_{11} = R_1''(Q_1) - C''(Q) \frac{\partial Q}{\partial Q_1} < 0$$

$$\Pi_{22} = R_2''(Q_2) - C''(Q) \frac{\partial Q}{\partial Q_2} < 0$$

$$\Pi_{12} = \Pi_{21} = \frac{\partial}{\partial Q_1} [R_2'(Q_2) - C'(Q)] = -C''(Q) < 0$$

- SOSC is satisfied when
1. $R_1'' - C'' < 0$
 2. $R_2'' - C'' < 0$
 3. $\Pi_{11}\Pi_{22} - \Pi_{12}^2 = [R_1'' - C''] [R_2'' - C''] - (-C'''(Q))^2 > 0$

ex. $P_1 = 22 - 2Q_1$, $P_2 = 10 - 0.5Q_2$ and $TC = 2Q + 5$.
Find Q_1, Q_2, P_1 and P_2 that maximize firm's profit.

$$\begin{aligned} \max_{Q_1, Q_2} \Pi(Q_1, Q_2) &= (22 - 2Q_1)Q_1 + (10 - 0.5Q_2) - (2Q + 5) \\ &= 22Q_1 - 2Q_1^2 + 10Q_2 - 0.5Q_2^2 - 2Q_1 - 2Q_2 - 5 \end{aligned}$$

$$\underline{FONC} \quad \frac{\partial \Pi}{\partial Q_1} = 20 - 4Q_1^* = 0 \Rightarrow Q_1^* = 5$$

$$\frac{\partial \Pi}{\partial Q_2} = 8 - Q_2^* = 0 \Rightarrow Q_2^* = 8$$

$$\underline{SOSC} \quad \left. \begin{array}{l} \Pi_{11} = -4 \\ \Pi_{22} = -1 \\ \Pi_{12} = \Pi_{21} = 0 \end{array} \right\} H = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}, |H_1| < 0, |H_2| > 0 \Rightarrow d^2\Pi < 0 \text{ max}$$

(Application: Input decision of a firm)

Let $Q = f(K, L) = 5K^{0.5}L^{0.25}$, $P = 4$, $w = 10$, $r = 5$. Find K^* , L^* max profit.

$$\Pi = P \cdot Q - TC$$

$$= 4(5K^{0.5}L^{0.25}) - (10K + 5L)$$

$$\max_{K, L} \Pi(K, L)$$

$$\underline{FONC} \quad \left. \begin{array}{l} \frac{\partial \Pi}{\partial K} = 10K^{-0.5}L^{0.25} - 5 = 0 \Rightarrow 2K^{-0.5}L^{0.25} - 1 = 0 \\ \frac{\partial \Pi}{\partial L} = 5K^{-0.5}L^{-0.75} - 10 = 0 \Rightarrow K^{-0.5}L^{-0.75} - 2 = 0 \end{array} \right\} \begin{array}{l} K^* = 4 \\ L^* = 1 \end{array}$$

$$\underline{SOSC} \quad \Pi_{KK} = -5K^{-1.5}L^{0.25} \Rightarrow \Pi_{KK}(K^* = 4, L^* = 1) = -\frac{5}{8} < 0$$

$$\Pi_{LL} = -3.75K^{0.5}L^{-1.75} \Rightarrow \Pi_{LL}(K^* = 4, L^* = 1) = -\frac{15}{2} < 0$$

$$\Pi_{KL} = 2.5K^{-0.5}L^{-0.75} \Rightarrow \Pi_{KL}(K^* = 4, L^* = 1) = -\frac{5}{4}$$

$$H = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} \\ \frac{5}{4} & -\frac{15}{2} \end{bmatrix} \quad |H_1| < 0, |H_2| > 0 \therefore \Pi(4, 1) \text{ is max.}$$

4 Comparative-Static Aspects of Optimization

ex. In PC, P_1, P_2 are exogenous

$$R = P_1Q_1 + P_2Q_2$$

$$C = 2Q_1^2 + Q_1Q_2 + Q_2^2$$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2) = P_1Q_1 + P_2Q_2 - (2Q_1^2 + Q_1Q_2 + Q_2^2)$$

$$\text{FONC} \quad \left. \begin{aligned} \frac{\partial \Pi}{\partial Q_1} = P_1 - 4Q_1^2 - Q_2 = 0 \\ \frac{\partial \Pi}{\partial Q_2} = P_2 - 4Q_2^2 - Q_1 = 0 \end{aligned} \right\} \begin{aligned} 4Q_1 + Q_2 = P_1 \\ Q_1 + 4Q_2 = P_2 \end{aligned}$$

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Rightarrow Q_1^* = \frac{4P_1 - P_2}{15}, \quad Q_2^* = \frac{P_2 - 4P_1}{15}$$

SOSC

$$\frac{\partial Q_1^*}{\partial P_1} = \frac{4}{15} > 0 \quad \frac{\partial Q_2^*}{\partial P_1} = \frac{-4}{15} < 0$$

$$\frac{\partial Q_1^*}{\partial P_2} = \frac{-1}{15} < 0 \quad \frac{\partial Q_2^*}{\partial P_2} = \frac{1}{15} > 0$$

ex. Consider $R = P \cdot Q(K, L)$ where $P, w,$ and r are exogenous. Find $\frac{\partial L^*}{\partial P}, \frac{\partial K^*}{\partial P}$.
Given, $C = wL + rK$

$$\max_{K, L} \Pi(K, L) = P \cdot Q(K, L) - (wL + rK)$$

$$\text{FONC} \quad \left. \begin{aligned} \Pi_K = P \cdot Q_K - r = 0 \\ \Pi_L = P \cdot Q_L - w = 0 \end{aligned} \right\} \frac{\partial K}{\partial L} = \frac{r}{w}$$

$$\Rightarrow K^* = K^*(P, w, r), \quad L^* = L^*(P, w, r)$$

If P changes, while w and r remain constant; what will happen to K^*, L^* ?

$$K^* = K^*(P) = K^*(P, \bar{w}, \bar{r})$$

$$L^* = L^*(P) = L^*(P, \bar{w}, \bar{r})$$

For FONC:

$$\Pi_K = P \cdot Q_K(K^*(P), L^*(P)) - r = 0$$

$$\Pi_L = P \cdot Q_L(K^*(P), L^*(P)) - w = 0$$

$$\begin{aligned}
\frac{d\Pi_K}{dP} &= \frac{d}{dP}[P \cdot Q_K(K^*(P), L^*(P)) - r] \\
&= P \cdot \frac{d}{dP}Q_K(K^*(P), L^*(P)) + Q_K(K^*(P), L^*(P))\frac{dP}{dP} - 0 \\
&\quad [\text{Note: } r \text{ is constant}] \\
&= P \cdot (Q_{KK} \cdot \frac{\partial K^*}{\partial P} + Q_{LK} \cdot \frac{\partial L^*}{\partial P}) + Q_K
\end{aligned}$$

$$\therefore \frac{d\Pi_K}{dP} \Rightarrow P \cdot Q_{KK} \cdot \frac{dK^*}{dP} + P \cdot Q_{LK} \cdot \frac{dL^*}{dP} = -Q_K$$

$$\frac{d\Pi_L}{dP} \Rightarrow P \cdot Q_{KL} \cdot \frac{dK^*}{dP} + P \cdot Q_{LL} \cdot \frac{dL^*}{dP} = -Q_L$$

$$\begin{bmatrix} P \cdot Q_{KK} & P \cdot Q_{LK} \\ P \cdot Q_{KL} & P \cdot Q_{LL} \end{bmatrix} \begin{bmatrix} \frac{dK^*}{dP} \\ \frac{dL^*}{dP} \end{bmatrix} = \begin{bmatrix} -Q_K \\ -Q_L \end{bmatrix}$$

$$\frac{dK^*}{dP} = \frac{-P \cdot Q_K Q_{LL} + P \cdot Q_L Q_{LK}}{P^2(Q_{KK}Q_{LL} - Q_{LK}^2)} > 0 \text{ if } Q_{LK} > 0$$

$$\frac{dL^*}{dP} = \frac{-P \cdot Q_L Q_{KK} + P \cdot Q_K Q_{LK}}{P^2(Q_{KK}Q_{LL} - Q_{LK}^2)} > 0 \text{ if } Q_{LK} > 0$$

\therefore L and K are complement

5 Multivariable Optimization

Let $z = f(x_1, x_2, x_3)$

$$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

FONC $dz = 0 \Leftrightarrow f_1 = f_2 = f_3 = 0$

SOSC $d^2z = \frac{\partial}{\partial x_1}(dz)dx_1 + \frac{\partial}{\partial x_2}(dz)dx_2 + \frac{\partial}{\partial x_3}(dz)dx_3$

⋮

$$= \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix} \underbrace{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}}_{\text{Hessian Matrix}} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

	$d^2z < 0$ (max)	$d^2z > 0$ (min)
$ H_1 = f_{11}$	$ H_1 < 0$	$ H_1 > 0$
$ H_2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$	$ H_2 > 0$	$ H_2 > 0$
$ H_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$	$ H_3 < 0$	$ H_3 > 0$
$\left. \begin{array}{l} H_1 = f_{11} \\ H_2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \\ H_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \end{array} \right\}$	<p>”negative definite”</p> <p>\Rightarrow all principal minor alternate its sign</p>	<p>”positive definite”</p> <p>\Rightarrow all principal minor must be positive</p>

Summary

	max	min
FONC	$f_1 = f_2 = \dots = f_n = 0$	$f_1 = f_2 = \dots = f_n = 0$
SOSC	$ H_1 < 0, H_2 > 0$ $ H_3 < 0, \dots$ or $(-1)^i H_i > 0$ for $i = 1, 2, \dots, n$	$ H_1 > 0, H_2 > 0$ $, H_3 > 0, \dots$ or $ H_i > 0$ for $i = 1, 2, \dots, n$

ex. $y = x_1^2 + 6x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3$

$$\frac{\partial y}{\partial x_1} = 2x_1 - 2x_2$$

$$\frac{\partial y}{\partial x_2} = 12x_2 - 2x_1 - 4x_3$$

$$\frac{\partial y}{\partial x_3} = 6x_3 - 4x_2$$

$$\left. \begin{array}{l} f_{11} = 2 \\ f_{22} = 12 \\ f_{33} = 6 \\ f_{12} = -2 \\ f_{13} = 0 \\ f_{21} = -2 \\ f_{23} = -4 \\ f_{31} = 0 \\ f_{32} = -4 \end{array} \right\} H = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 12 & -4 \\ 0 & -4 & 6 \end{bmatrix}$$

$$|H_1| = 2 > 0$$

$$|H_2| = 20 > 0$$

$$|H_3| = 88 > 0$$

$\therefore d^2y > 0$ positive definite (min)

(Application: Multimarket Monopoly)

$$\begin{aligned} \text{Let } P_1 &= 63 - 4Q_1 \\ P_2 &= 105 - 5Q_2 \\ P_3 &= 75 - Q_3 \\ TC &= 15Q + 20 \end{aligned}$$

Find Q_i and P_i for $i = 1, 2, 3$ that max Π

$$\Rightarrow \max_{Q_1, Q_2, Q_3} \Pi(Q_1, Q_2, Q_3)$$

$$= (63 - 4Q_1)Q_1 + (105 - 5Q_2)Q_2 + (75 - Q_3)Q_3 - [15(Q_1 + Q_2 + Q_3) + 20]$$

$$\begin{aligned} \underline{FONC} \quad \Pi_1 &= 48 - 8Q_1 \Rightarrow Q_1^* = 6 \\ \Pi_2 &= 90 - 10Q_2 \Rightarrow Q_2^* = 9 \\ \Pi_3 &= 60 - 20Q_3 \Rightarrow Q_3^* = 30 \end{aligned}$$

$$\underline{SOSC} \quad \left. \begin{array}{l} \Pi_{11} = -8 \\ \Pi_{22} = -10 \\ \Pi_{33} = -2 \\ \Pi_{12} = \Pi_{13} = 0 \\ \Pi_{21} = \Pi_{23} = 0 \\ \Pi_{31} = \Pi_{33} = 0 \end{array} \right\} H = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$|H_1| = -8 < 0$$

$$|H_2| = 80 > 0$$

$$|H_3| = -160 > 0$$

$\therefore d^2z < 0$ negative definite

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