CHAPTER 4

Basic Matrix Algebra and Applications

Topics: Economic Applications of Matrix Algebra

Outline:

- How to use matrix to solve system of simultaneous equations
- Partial market equilibrium
- The effect of specific tax on market equilibrium
- Simple Macroeconomic Model
- IS – LM

How to use matrix to solve system of simultaneous equations

When we have the system of simultaneous equations:

1.) Identify endogenous variables, exogenous variables, parameters:

This is a system of m linear equations with n endogenous variables:

n Vars

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m
\end{aligned}
$$

m eqs

2.) Reduce the system of equations to have number of equations equal to number of endogenous variables and to have only necessary information

That is, we need to have:

$$\textit{number of equations} = \textit{number of endogenous variables}$$

3.) Rearrange every equations in the system so that all endogenous variables are on one side and all other exogenous variables on the other side

From:

$$
\begin{aligned}
ax_1 &= k - bx_2 \\
dx_2 &= h - cx_1
\end{aligned}
$$

To:

$$
\begin{aligned}
ax_1 + bx_2 &= k \\
cx_1 + dx_2 &= h
\end{aligned}
$$

4.) Rewrite the system into matrix form:

$$
\begin{aligned}
ax_1 + bx_2 &= k \\
cx_1 + dx_2 &= h
\end{aligned}
$$

$$\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} k \\ h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} k \\ h \end{bmatrix}_{2\times 1}$$

$$A = \begin{bmatrix} o & o & o \\ o & o & o \\ o & o & o \end{bmatrix}$$

$$\boxed{A_{n\times n}x_{n\times 1}=d_{n\times 1}}$$

A is matrix of parameters that are in front of endogenous variables.

x is column vector of endogenous variables to solve for its final values as a function of exogenous variables and parameters

d is column vector of parameters that don't multiply with endogenous variables and exogenous variables

5.) Check that det A must not equal to 0, so that A^{-1} exists.

$$|A| \neq 0$$

,where det A can be computed by:

- If a square matrix A is a 2 by 2 matrix, then det A is equal to:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

- If a square matrix A is a 3 by 3 matrix, then det A is equal to:

$$|A|$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

- n^{th}-order determinant by Laplace Expansion

Let A be a matrix in $R^{n\times n}$. The determinant of A or det A is given by $|A|$.

$$|A| = \begin{cases} \sum_{j=1}^{n} a_{ij}C_{ij} & i = 1,2,\ldots,n \text{ (expansion by i}^{\text{th}}\text{row)} \\ \sum_{i=1}^{n} a_{ij}C_{ij} & j = 1,2,\ldots,n \text{ (expansion by j}^{\text{th}}\text{column)} \end{cases}$$

We call C_{ij}, Cofactor of $element\ a_{ij}$.

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$The\ minor\ of\ element\ a_{ij}, M_{ij}$, is the determinant of the matrix resulting from deleting row i and column j of matrix $A_{n\times n}$. In other words, a "minor" is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix.

Note: If determinant is found by using Cofactor for elements from the WRONG row or column, the determinant will be zero!!!!!

$$\begin{vmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \qquad |C_{21}| = -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3 \qquad |C_{22}| = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10 \qquad |C_{23}| = -\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$|A| = 4(-3) + 1(10) + 2(1) = 0$$

6.) Use inverse matrix **or** Cramer's rule to find the solution of the system of simultaneous equations:

6.1) Inverse matrix:

$$Ax = d$$
$$A^{-1}Ax = A^{-1}d$$
$$Ix = A^{-1}d$$

$$\boxed{x = A^{-1}d}$$

,where we can find A^{-1} by

$$A^{-1} = \frac{1}{|A|} \cdot adjA$$

$$adjA = [C_{ij}]^T$$

6.2) Cramer's rule:

$$x_k = \frac{|A_k|}{|A|}$$

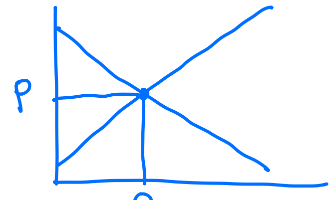
$|A_k|$ is the determinant of matrix A when replacing column k of matrix A by vector *d.*

Partial market equilibrium

Demand: $Q^D = a - bP$

Supply: $Q^S = -c + dP$

Equilibrium: $Q^D = Q^S$



$Q^D = Q^S$

1.) Identify endogenous variables, exogenous variables, parameters:

Endogenous variables:……… Q^D, Q^S, P ………………………

2.) Reduce the system of equations to have number of equations equal to number of endogenous variables

How many equations will we be working with? ………………3………………………

3.) Rearrange every equations in the system so that all endogenous variables are on one side and all other exogenous variables on the other side

Demand: $Q^D = a - bP$ $\Rightarrow$ $Q^D + bp = a$

$Q^D + 0Q^S + bp = a$

Supply: $Q^S = -c + dP$

Equilibrium: $Q^D = Q^S$

$Ax = d = \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}$

Demand: $\underline{1}\, Q^D \underline{+0}\, Q^S \underline{+b}\, P = a$

Supply: $\underline{0}\, Q^D \underline{+1}\, Q^S \underline{-d}\, P = -c$

Equilibrium: $\underline{1}\, Q^D \underline{-1}\, Q^S \underline{+0}\, P = 0$

$\Rightarrow Ax = d$

$A^{-1}Ax = d$

$x = A^{-1}d$

column vecter

4.) Rewrite the system into matrix form:

x, d, A

$1Q^D + 0Q^S + bp$

Ceffi in Demand f^n: row 1 →
Ceffi in supply f^n row 2 →
equi an : row 3 →

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} Q^D \\ Q^S \\ p \end{bmatrix} = \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}$$

$\}\ A x = d$

col 1 ceffi of Q^D

col 2 Ceffi of Q^S

col 3 Ceffi of p

$A^{-1}Ax = A^{-1}d$

$Ix = A^{-1}d$

$\boxed{x = A^{-1}d}$

Laplace's expansion by ① Pick any row or any column
② det A is the summation of the multiplication of the element of matrix A in that row or column with cofactor of each element.

5.) *Check that det A must not equal to 0, so that A^{-1} exists.*

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$

Assignment 1 : due Mar 17
Tutor session : Mar 5, 4.30pm

$$A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}$$

$\det A = |A| = +(1\times 1\times 0) + (0\times -d\times 1) + (b\times 0\times -1) - (1\times 1\times b) - (-1\times -d\times 1) - (0\times 0\times 0)$

$\det A = |A| = -b - d \neq 0$

if $|A| \neq 0$, A is a nonsingular, A has A^{-1}.

6.) *Use inverse matrix **or** Cramer's rule to find the solution of the system of simultaneous equations:*

$A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}$

6.1) *Inverse matrix:*

$$\boxed{A^{-1} = \frac{1}{|A|}\cdot adjA}$$

$[C_{ij}]$ is called a cofactor matrix of matrix A

$$adjA = [C_{ij}]^T$$

$C_{ij} = (-1)^{i+j} M_{ij}$

M_{ij} = the det of A after deleting row i & col j

Col 1 Col 2 Col 3

row 1, 2, 3

M_{11}

$$C = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} \end{bmatrix}_{3\times 3}$$

$M_{11} = 0 - d$

$$C = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 0 & b \\ -1 & 0 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & b \\ 1 & 0 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 0 & b \\ 1 & -d \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & b \\ 0 & -d \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

M_{11}

$$C=\begin{bmatrix}(-1)^{1+1}(-d) & (-1)^{1+2}d & (-1)^{1+3}(-1)\\ (-1)^{2+1}b & (-1)^{2+2}(-b) & (-1)^{2+3}(-1)\\ (-1)^{3+1}(-b) & (-1)^{3+2}(-d) & (-1)^{3+3}(1)\end{bmatrix}$$

cofactor A = $C=\begin{bmatrix}-d & -d & -1\\ -b & -b & 1\\ -b & d & 1\end{bmatrix}$

$$adjA=C^T=\begin{bmatrix}-d & -b & -b\\ -d & -b & d\\ -1 & 1 & 1\end{bmatrix}$$

$$\boxed{A^{-1}}=\frac{1}{detA}adjA=\boxed{\frac{1}{-b-d}\begin{bmatrix}-d & -b & -b\\ -d & -b & d\\ -1 & 1 & 1\end{bmatrix}}$$

Therefore,

$$\frac{-ad+bc-b(0)}{-b-d}$$

$$x=A^{-1}d$$

A^{-1} d

$$\begin{bmatrix}Q^D\\ Q^S\\ P\end{bmatrix}=\boxed{\frac{1}{-b-d}\begin{bmatrix}-d & -b & -b\\ -d & -b & d\\ -1 & 1 & 1\end{bmatrix}}\boxed{\begin{bmatrix}a\\ -c\\ 0\end{bmatrix}}=\begin{bmatrix}x_{11}\\ x_{21}\\ x_{31}\end{bmatrix}$$

column vector

$$\begin{bmatrix}Q^D\\ Q^S\\ P\end{bmatrix}=\begin{bmatrix}\frac{ad-bc}{b+d}\\ \frac{ad-bc}{b+d}\\ \frac{a+c}{b+d}\end{bmatrix}$$

6.2) Use Cramer's rule: $x_k=\frac{|A_k|}{|A|}$

$x_k = x_{kj}$

$x_1=\frac{|A_1|}{|A|}$, $x_2=\frac{|A_2|}{|A|}$

$x_3=\frac{|A_3|}{|A|}$

$x_{11}=x_1$

$x_{21}=x_2$

$x_{31}=x_3$

$$\begin{bmatrix}Q^D\\ Q^S\\ P\end{bmatrix}=\begin{bmatrix}\frac{|A_1|}{|A|}\\ \frac{|A_2|}{|A|}\\ \frac{|A_3|}{|A|}\end{bmatrix}$$

Containing exogenous var & parameter

Replace Col 1 with vector d

Col 1 originally contains coeffi of Q^D in matrix A

$$x_1 = x_{11} = Q^D = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} a & 0 & b \\ -c & 1 & -d \\ 0 & -1 & 0 \end{vmatrix}}{-b-d}$$

$$= \frac{bc-ad}{-b-d}$$

Col 2 is originally coeffi of Q^S

$$Q^S = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & a & b \\ 0 & -c & -d \\ 1 & 0 & 0 \end{vmatrix}}{-b-d}$$

$$P = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & -c \\ 1 & -1 & 0 \end{vmatrix}}{-b-d}$$

☺ If we assume $Q^D = Q^S = Q$,

From the model:

Demand:	$Q^D = a - bP$
Supply:	$Q^S = -c + dP$
Equilibrium:	$\boxed{Q^D = Q^S}$

3 endo 3 eqs.

If we assume $\boxed{Q^D = Q^S = Q}$, then we can reduce the system of simultaneous equations down to:

Demand:	$Q = a - bP$
Supply:	$Q = -c + dP$

Rearrange & Rewrite into matrix form:

Demand:	$Q + bP = a$
Supply:	$Q - dP = -c$

$$\begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix}\begin{bmatrix} Q \\ P \end{bmatrix} = \boxed{\begin{bmatrix} a \\ -c \end{bmatrix}}$$

$$|A| = -d - b$$

$$A x = d$$

$$x = A^{-1} d \Rightarrow \begin{bmatrix} Q \\ P \end{bmatrix} = \underbrace{\boxed{\frac{1}{-d-b}\begin{bmatrix} -d & -b \\ -1 & 1 \end{bmatrix}}}_{A^{-1}}\begin{bmatrix} a \\ -c \end{bmatrix}$$

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \dfrac{\begin{vmatrix} a & b \\ -c & -d \end{vmatrix}}{-b-d} \\ \dfrac{\begin{vmatrix} 1 & a \\ 1 & -c \end{vmatrix}}{-b-d} \end{bmatrix}$$

$|A_1|$, $|A_2|$

$Q = \dfrac{|A_1|}{|A|}$, $P = \dfrac{|A_2|}{|A|}$

$= \begin{bmatrix} \dfrac{-ad+bc}{-b-d} \\ \dfrac{-c-a}{-b-d} \end{bmatrix}$

☺ If we collect specific tax from supplier t baht per unit

From the model:

Demand: $Q^D = a - bP^D$

Supply: $Q^S = -c + dP^S$

Equilibrium: $Q^D = Q^S$

$P^S = P^D - t$

Q, P, $P - t$, Q

4 eqs, 4 endo $\Rightarrow Q^D, Q^S, P^D, P^S$

$\underset{A}{\underset{4\times4}{[\quad]}}\underset{x}{[\quad]} = \underset{d}{[\quad]}$

If we assume $Q^D = Q^S = Q$ and $P^D = P$, then we can reduce the system of simultaneous equations down to:

equilibrium price

equilibrium Quantity

Demand: $Q = a - bP$

Supply: $Q = -c + dP$ $-dt$

Rearrange & Rewrite into matrix form:

Demand: $Q + bP = a$

Supply: $Q - dP = -c$ $-dt$

no tax

$Q + bP = a$

$Q - dP = -c$

$$\begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix}\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} a \\ -c-dt \end{bmatrix} \rightarrow A^{-1}$$

↓ use Cramer's rule

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \dfrac{\begin{vmatrix} a & b \\ -c-dt & -d \end{vmatrix}}{-b-d} \\ \dfrac{\begin{vmatrix} 1 & a \\ 1 & -c-dt \end{vmatrix}}{-b-d} \end{bmatrix}$$

Exercise:

1.)

$Q^D = 24 - 4P$

$Q^S = 13P - 27$ $\quad (P^* = 3, Q^* = 12)$

$Q^D = Q^S$

2.) Consider a simple model in which two commodities are related to each other

Good 1	Good 2
$Q_{d1} = Q_{s1}$	$Q_{d2} = Q_{s2}$
$Q_{d1} = a_0 + a_1P_1 + a_2P_2$	$Q_{d2} = c_0 + c_1P_1 + c_2P_2$
$Q_{s1} = b_0 + b_1P_1 + b_2P_2$	$Q_{s2} = d_0 + d_1P_1 + d_2P_2$

Find the equilibrium prices and quantities in good 1 and good 2

Simple Macroeconomic Model

Consider simple Keynesian crossing model:

(1.) $Y = C + I + G$

(2.) $C = a + bY_d$

(3.) $Y_d = Y - T$

(4.) $I = I_0$

(5.) $G = G_0$

Can we use all 5 equations ?

$A_{5\times5}x = d$

no!

Identify endogenous variables, exogenous variables, parameters:

Endogenous variables:……Y, C, Yd……………………

Reduce the system of equations to have number of equations equal to number of endogenous variables

How many equations will we be working with? ……3……………………

$$\left\{\begin{aligned} Y &= C + I_0 + G_0 \\ C &= a + bY^d \\ Y^d &= Y - T \end{aligned}\right\}$$

$Ax = d$

Rearrange every equations in the system so that all endogenous variables are on one side and all other exogenous variables on the other side

$$Y = C + I_0 + G_0$$
$$C = a + bY^d$$
$$Y^d = Y - T$$

$$\underline{1}\ Y \ \underline{-1}\ C \ \underline{+0}\ Y^d = \underline{I_0 + G_0}$$
$$\underline{0}\ Y \ \underline{+1}\ C \ \underline{-b}\ Y^d = \underline{a}$$
$$\underline{-1}\ Y \ \underline{+0}\ C \ \underline{+1}\ Y^d = \underline{-T}$$

Rewrite the system into matrix form:

coeff of Y, of C, of Y^d

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -b \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ Y^d \end{bmatrix} = \begin{bmatrix} I_0+G_0 \\ a \\ -T \end{bmatrix}$$

$$Ax = d$$

$$Y^* = \frac{\begin{vmatrix} \underline{I_0+G_0} & -1 & 0 \\ \underline{a} & 1 & -b \\ \underline{-T} & 0 & 1 \end{vmatrix}}{1-b}$$

$$: \quad x_1 = x_{11} = \frac{|A_1|}{|A|}$$

$$C^* = \frac{\begin{vmatrix} 1 & \underline{I_0+G_0} & 0 \\ 0 & \underline{a} & -b \\ -1 & \underline{-T} & 1 \end{vmatrix}}{1-b} = \frac{a + b(I_0+G_0) - bT}{1-b}$$

$$Y^{D*} = \frac{\begin{vmatrix} 1 & -1 & I_0 + G_0 \\ 0 & 1 & a \\ -1 & 0 & -T \end{vmatrix}}{1-b}$$

Note: We can also work with the system with two equations:

From

$Y = C + I_0 + G_0$

$C = a + bY^d$

$Y^d = Y - T$

To

$Y = C + I_0 + G_0$

$C = a + bY - bT$

IS – LM

The Good Market:

1 $Y = C + I + G$
2 $C = C_0 + b(1-t)Y$
3 $I = I_0 - er$
4 $G = G_0$

3 eqs → IS

$(1-b(1-t))Y + er = C_0 + I_0 + G_0$ IS

Y & r
2 endo

The Money Market:

5 $M_d = fY - \beta r$
6 $M_s = M_0$
7 $M_d = M_s$

1 eq → LM

$fY - \beta r = M_0$ LM

Identify endogenous variables, exogenous variables, parameters:

Endogenous variables: $Y, C, I,$ ...M_d, r.........

Reduce the system of equations to have number of equations equal to number of endogenous variables

How many equations will we be working with? ......4 equations......

$$Y = C + I + G_0$$
$$C = C_0 + b(1-t)Y$$
$$I = I_0 - er$$
$$M_0 = fY - \beta r$$

4 endo var
Y, C, I, r

Rearrange

$$\boxed{Y - C - I + 0r} = G_0$$
$$-b(1-t)Y + C + 0I + 0r = C_0$$
$$0Y + 0C + I + er = I_0$$
$$fY + oC + 0I - \beta r = M_0$$

Rewrite the system into matrix form:

$$\begin{array}{c} A \\ \begin{bmatrix} 1 & -1 & -1 & 0 \\ -b(1-t) & 1 & 0 & 0 \\ 0 & 0 & 1 & e \\ f & 0 & 0 & -\beta \end{bmatrix} \\ 4\times4 \end{array} \begin{array}{c} x \\ \begin{bmatrix} Y \\ C \\ I \\ r \end{bmatrix} \end{array} = \begin{array}{c} d \\ \begin{bmatrix} G_0 \\ C_0 \\ I_0 \\ M_0 \end{bmatrix} \end{array}$$

A^{-1}

Cramer's rule

Tips: In this case, it will be easier if you use Cramer's rule for the solution and you will need to use Laplace's expansion to compute the determinants

Alternatively, we can use matrix algebra to solve the system of two equations: IS-LM

$$(1-b(1-t))Y + er = C_0 + I_0 + G_0$$

$$fY - \beta r = M_0$$

$$\begin{array}{c} A \\ \begin{bmatrix} 1-b(1-t) & e \\ f & -\beta \end{bmatrix} \end{array} \begin{array}{c} x \\ \begin{bmatrix} Y \\ r \end{bmatrix} \\ 2\times2 \end{array} = \begin{array}{c} d \\ \begin{bmatrix} C_0+I_0+G_0 \\ M_0 \end{bmatrix} \end{array}$$

a_{11}, a_{12}, a_{21}, a_{22}

$$|A| = -\beta(1-b(1-t)) - fe \neq 0$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A, \quad \operatorname{adj} A = C^T$$

$C = [C_{ij}]$, $C_{ij} = (-1)^{i+j} M_{ij}$

$$A = \begin{bmatrix} 1-b(1-t) & e \\ f & -\beta \end{bmatrix}$$

M_{ij} is det of matrix A after deleting row i & col j

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} -\beta & f \\ e & 1-b(1-t) \end{bmatrix}$$

$$C = \begin{bmatrix} -\beta & -f \\ -e & 1-b(1-t) \end{bmatrix}$$

$$A^{-1} = \frac{1}{-\beta(1-b(1-t)) - fe} \begin{bmatrix} -\beta & -e \\ -f & 1-b(1-t) \end{bmatrix}$$

$$\begin{bmatrix} Y \\ r \end{bmatrix} = A^{-1} d$$

$$\begin{bmatrix} Y \\ r \end{bmatrix} = \frac{1}{-\beta(1-b(1-t)) - fe} \begin{bmatrix} -\beta & -e \\ -f & 1-b(1-t) \end{bmatrix} \begin{bmatrix} C_0 + I_0 + G_0 \\ M_0 \end{bmatrix}$$

$$Y = \frac{1}{-\beta(1-b(1-t)) - fe} \times -\beta(C_0 + I_0 + G_0) - eM_0$$

$$r = \frac{1}{-\beta(1-b(1-t)) - fe} \times -f(C_0 + I_0 + G_0) + M_0(1-b(1-t))$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det A = ad - bc \neq 0$$

$$M_{11} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad M_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Rightarrow M = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad M_{22} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} M_{11} = d \qquad C_{12} = (-1)^{1+2} M_{12} = -1 \times (c) = -c$$

$$\Rightarrow C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$C_{21} = (-1)^{2+1} M_{21} = -1 \times b = -b \qquad C_{22} = (-1)^{2+2} M_{22} = a$$

$$\text{Adj } A = C^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \boxed{\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$