Exercise 7

Integrals: Indefinite Integral/Integration by the u-Substitution/Area/Definite Integral/FTC

1. Evaluate the given indefinite integrals (these may require to use the substitution rule).

   (a) $\int \frac{(x-3)^2}{\sqrt{x}}\ dx$ Ans: $(2/5)x^{5/2} - 4x^{3/2} + 18x^{1/2} + C$

   (b) $\int \left[e^x + \frac{1}{x} - \sin(x) + 3\sinh(x) + \pi^2 + \csc^2(x) - \sec(x)\tan(x)\right]\ dx$
   Ans: $e^x + \ln|x| + \cos(x) + 3\cosh(x) + \pi^2 x - \cot^2(x) - \sec(x) + C$

   (c) $\int \sqrt{2+3x}\ dx$ Ans: $\frac{2}{9}(2+3x)^{3/2} + C$

   (d) $\int \frac{2\cos(\ln(x))}{x}\ dx$ Ans: $2\sin(\ln(x)) + C$

   (e) $\int e^{5x}(1-e^{5x})^{1/3}\ dx$ Ans: $-\frac{3}{20}(1-e^{5x})^{4/3} + C$

   (f) $\int \frac{x}{9+4x^2}\ dx$ Ans: $\frac{1}{8}\ln|9+4x^2| + C$

   (g) $\int \frac{1}{9+4x^2}\ dx$ Ans: $\frac{1}{6}\tan^{-1}(2x/3) + C$

   (h) $\int \frac{1-x}{\sqrt{3-x^2}}\ dx$ Ans: $\sin^{-1}(\frac{x}{\sqrt{3}}) + \sqrt{3-x^2} + C$

   (i) $\int \left[\sin^3(x)\cos(x) + \frac{\sec^2(x)}{\sqrt{\tan(x)}}\right]\ dx$ Ans: $\frac{1}{4}\sin^4(x) + 2\sqrt{\tan(x)} + C$

   (j) $\int [\tan(\pi x) + \sec(2\pi x)\ ]\,dx$ Ans: $-\frac{1}{\pi}\ln|\cos(\pi x)| + \frac{1}{2\pi}\ln|\sec(2\pi x) + \tan(2\pi x)|$
   Hint: To integrate $\sec(2\pi x)$, let $u = \sec(2\pi x) + \tan(2\pi x)$

   (k) $\int \frac{1}{[1+x^2][\tan^{-1}(x)]}\ dx$ Ans: $\ln|\tan^{-1}(x)| + C$

   (l) $\int 2x^3\sqrt{1+x^2}\ dx$ Ans: $\frac{2}{5}(x^2+1)^{5/2} - \frac{2}{3}(x^2+1)^{3/2} + C$

   (m) $\int e^{x+e^x}\ dx$ Ans: $e^{e^x} + C$

2. Determine the function $f(x)$ such that $f''(x) = (1+x)^4$, $f'(0) = 0$ and $f(0) = 0$.
   Ans: $f(x) = \frac{(x+1)^6}{30} - \frac{x}{5} - \frac{1}{30}$.

3. (Optional) Recall that the area A under the graph of a non-negative function $f(x)$ on a closed interval $[a,b]$ can be obtained from

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\triangle x_k,$$

   where $x_k^* = a + k\triangle x$ and $\triangle x_k = \triangle x = \frac{b-a}{n}$.

   (i) Use above equation to find the area A when $f(x) = x^2 + 2$, $a = 0$ and $b = 2$.

   (ii) Evaluate $\int_0^2 x^2 + 2\ dx$ by using *the Fundamental Theorem of Calculus* and compare to (i).
   Ans: $\frac{20}{3}$

4. (a) Find the area between the function $f(x) = |x-2|$ and the x-axis on the interval $[0,2]$.

(b) Find the area between the function $f(x) = x - 2$ and the x-axis on the interval $[0, 2]$.

(c) Evaluate $\int_0^2 |x-2|\ dx$ and $\int_0^2 x-2\ dx$.

Ans: (a) 2 (b) 2 (c) $2, -2$

5. Use the Fundamental Theorem of Calculus (derivative form)to find the indicated derivative.

(a) $\frac{d}{dx}\int_2^x \left[t^2e^t + \ln(|\sin(t)|)\right]\ dt$ Ans: $x^2e^x + \ln(|\sin(x)|)$

(b) $\frac{d}{dx}\int_{e^x}^{\pi} \ln(t)\tan(2t)\ dt$ Ans:$-e^x\ln(e^x)\tan(2e^x)$

(c) $\frac{d}{dx}\int_{\ln(x)}^{\sin(x)} \frac{1}{1+t^5}\ dt$ Ans: $-\frac{1}{x[1+(\ln(x))^5]} + \frac{\cos(x)}{1+[\sin(x)]^5}$

(d) $\frac{d}{dx}\int_{\pi}^{e^{\pi}} \ln(t)\ dt$ Ans: 0

6. Evaluate the following definite integrals.

(a) $\int_1^2 \left[3x^2 - 2x + 1 - \frac{1}{x^2} + \frac{1}{3x} + e^x\right]\ dx$ Ans: $\frac{9}{2} + \frac{1}{3}\ln(2) + e^2 + e$

(b) $\int_1^5 \frac{1}{1+2x}\ dx$ Ans: $\frac{1}{2}\ln(\frac{11}{3})$

(c) $\int_1^2 \left[\frac{x^2+1}{x} + \frac{x}{(x^2+1)^2}\right]\ dx$ Ans: $\frac{33}{20} + \ln(2)$

(d) $\int_0^{\frac{1}{2}} \sin^2(\pi x)\ dx$ Ans: $\frac{1}{4}$

(e) $\int_0^{\frac{\pi}{2}} \cos(x) + \cos^5(x)\sin(x)\ dx$ Ans: $\frac{7}{6}$

(f) $\int_{-1}^1 \left(\frac{x^5+x}{(x^4+4x^2+4)^3} + |x|\right)\ dx$ Ans: 1

(g) $\int_0^{\frac{3\pi}{2}} |\cos(x)|\ dx$ Ans: 3

(h) $\int_{\frac{1}{2}}^{e} \frac{[\ln(2x)]^5}{x}\ dx$ Ans: $\frac{1}{6}(\ln(2)+1)^6$

(i) $\int_0^{\frac{1}{\sqrt{2}}} \frac{t}{\sqrt{1-t^4}}\ dt$ Ans:$\frac{\pi}{12}$

(j) $\int_0^1 \frac{e^{3t}}{2-e^{3t}}\ dt$ Ans:$-\frac{1}{3}\ln(|2-e^3|)$

7. Let $f(x) = \begin{cases} -2, & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$. Evaluate $\int_{-2}^2 f(x)\ dx$ and $\int_{-2}^2 |f(x)|\ dx$. Ans:$-2, 6$