Solution: Quiz 6

1. Consider a relation
$$f = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \middle| \quad y = x^3 - 3x \right\}.$$
Identify x and y intercepts, symmetry, relative extrema, and concavity of f.
Sketch the curve of f.

Answer: Notice that f is a function and we can write it as $f(x) = x^3 - 3x$.

- **x-intercepts:** by setting $y = 0$, we have the x-intercepts are $(-3, 0), (0, 0), (3, 0)$. I.e.,
$$0 = x^3 - 3x = x(x^2 - 3) = x(x - \sqrt{3})(x + \sqrt{3}) \quad \Leftrightarrow \quad x = -3, 0, 3.$$
y-intercepts: setting $x = 0$ gives $y = 0^3 - 3 \cdot 0 = 0$ and the y-intercept is $(0, 0)$.
- **Symmetry:** From $f(-x) = (-x)^3 - 3(-x) = -[x^3 - 3x] = -f(x)$, f is **symmetric about the origin**. Note that since $f(x) \neq f(-x)$, $f(x)$ is **not** symmetric about the y-axis. Also, since, for $(x, y) \in f$, $(x, -y) \notin f$ (i.e. f is a function), it is **not** symmetric about the x-axis.
- **Relative extremum:** Notice that
$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 0 \quad \Leftrightarrow \quad x = -1, 1.$$
So -1 and 1 are the critical values ($f'(x)$ is well-defined for any $x \in \mathbb{R}$). By using the Second Derivative test and from $f''(x) = 6x$, we can characterize critical numbers as follows:
  - For $x = -1$, $f''(-1) = -6 < 0$ implies that a relative maximum occurs at $x = -1$ and $f(-1) = -1 + 3 = 2$ is a relative maximum of f,
  - For $x = 1$, $f''(1) = 6 > 0$ implies that a relative minimum occurs at $x = 1$ and $f(1) = 1 - 3 = -2$ is a relative minimum of f.
- **Concavity and inflection points:** From $f''(x) = 6x$, $f''(x) = 0$ only when $x = 0$.
  - Concave down: $f''(x) < 0$ when $6x < 0$ or $x \in (-\infty, 0)$.
  - Concave up: $f''(x) > 0$ when $6x > 0$ or $x \in (0, \infty)$.

  The point of inflection is therefore $(0, f(0)) = (0, 0)$ because the curve changes the concavity at this point. The sketch of this curve is shown below.