Part I System of Linear Equations and Linear Algebra

Chapter 1 System of Linear Equations

In economic modeling, linear equations are encountered frequently due its ability to approximate complex systems despite its simplicity. Consult **Simon & Blume** [1994], pages 108-120 for examples. The analysis of system of linear equations will lead naturally to the use of matrices and, in a later chapter, the vector spaces. The matrix theory is also essential in establishing the optimal conditions in the classical optimization to be discussed in Part IV.

In this chapter, we will not only learn how to solve a system of linear equations, but more importantly how to determine whether the system has a solution, and when it does, if it is unique. This is the question of the existence and uniqueness of a solution of a system of linear equations.

1.1 System of Linear Equations

Definition 1.1 A ***system of m linear equations with n variables*** is given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} is the ***coefficient*** of ***variable*** x_j in the i^{th} equation, b_i is a fixed real number, called the ***right hand side*** (RHS) of the i^{th}equation, for each $i = 1,2,\dots,m,$ and $j = 1,2,\dots,n$. A ***solution*** to the above system of linear equations is an assignment of n values to the n variables so that all m equations are simultaneously satisfied.

Example Given a system of linear equations,

$$\begin{aligned} 3x_1 + 2x_2 &= 6 \\ 2x_1 - 4x_2 &= 12 \end{aligned}$$

The variables are x_1 and x_2. In the first equations, 3 and 2 are the coefficients of x_1 and x_2 respectively, with 6 being the RHS. A solution to the system of equations is given by $(x_1 = 3, x_2 = -1.5)$.

The obvious question is how to find a solution. But, more importantly, is there a solution in the first place? If there is, is the solution unique?

Example Leontief's Linear Model of Production (See **Simon & Blume** [1994], page 110, and **Gale** [1960].

1.2 Solving System of Linear Equations by Gaussian Elimination Method

Definition 1.2 Two systems of linear equations are ***equivalent*** if a solution to one system is also a solution to another. Thus, if a system of linear equations has no solution, unique solution, or multiple solutions, so does any of its equivalent systems.

Definition 1.3 Given a system of linear equations, the following operations, called ***elementary row operations***, will create an equivalent system.

a) multiply an equation by a nonzero constant,
b) interchange any pair of equations,
c) multiply an equation by a constant and add it to another.

The Gaussian Elimination method solves a system of linear equations by reducing it into a simpler one such that the first k equations have more and more ***leading zero coefficients*** in each subsequent equations, and the last $m-k$ equations have only zero coefficients. The final system will be:

$$\begin{array}{cccccccccc}
a'_{11}x_1 & +\cdots & \cdots & \cdots & \cdots & \cdots & \cdots + & a'_{1n}x_n & = & b'_1 \\
0 & \cdots & 0 & a'_{2j_2}x_{j_2} & +a'_{2j_2+1}x_{j_2+1} & +\cdots & \cdots + & a'_{2n}x_n & = & b'_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & a'_{kj_k+1}x_{j_k+1} & +\cdots+ & a'_{kn}x_n & = & b'_k \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & = & b'_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & = & b'_m
\end{array}$$

where $1 \le k \le m$, $1 < j_2 < j_3 < \cdots < j_k \le n$ and $a'_{11} \neq 0$, $a'_{rj_r} \neq 0$ for all $r = 2,3,\dots,k$. We will call a system of linear equations in this form as being in ***echelon form***. It is obvious that any system of linear equations is equivalent to its echelon form resulting from a series of elementary row operations.

Proposition 1.1 An echelon form of any system of linear equations can be obtained after a finite number of elementary row operations.

Proof Exercise. Note that it can be shown that the number of operations is in the order of n^2.❑

Definition 1.4 In the echelon form, we will call each of these first nonzero coefficients $a'_{11} \neq 0$, $a'_{rj_r} \neq 0$ for all $r = 2,3, \dots, k$, a ***pivot***. (See **Simon & Blume** [1994], page 132)

Example The system

$$\begin{aligned} 2x_1 - x_2 + 5x_3 &= 2 \\ 3x_3 &= 6 \end{aligned}$$

is in echelon form with coefficients 2 and 3 being pivots.

We can make the following conclusions:

Theorem 1.1 Given a system of linear equation in echelon form as in (1.1),

a) if $b'_r \neq 0$, for some $k+1 \leq r \leq m$, then this system of linear equations does not have a solution. We say that the system is ***inconsistent***, and
b) if $b'_r = 0$, for all $k+1 \leq r \leq m$, the system of linear equations is ***consistent*** and has a solution, and
   (b.*i*) if $k = n$, then the solution is ***unique***,
   (b.*ii*) if $k < n$, then there are infinite number of solutions.

Proof **a)** There are no values that we can assign to the variables and then multiply each of them with zero and the sum becomes $b'_r \neq 0$, $k+1 \leq r \leq m$.

b) If $k = n$, there are *n* equations with n variables. We will have pivots forming a perfect diagonal pattern. That is, we have $j_2 = 2, j_3 = 3, \dots, j_k = k = n$. In the last equation of the system, we have simply, $a'_{nn}x_n = b'_n$, and x_n can be uniquely determined as $a'_{nn} \neq 0$. With this value of x_n, we can use equation $n-1$ to uniquely determine x_{n-1}, and so on. Thus all the variables are uniquely determined.

If $k < n$, there are $n-k$ more variables than equations that have nonzero coefficients. Since each of k nonzero equations contains a pivot, it follows that there are

$n-k$ variables whose columns of coefficients do not contain a pivot. We can assign arbitrary values to these $n-k$ variables. The resulting system of linear equations will still be in echelon form with k variables and k equations, and thus will have a unique solution. Since the $n-k$ variables are assigned arbitrary values, there are infinite number of solutions.❑

Notes: 1. Why is the case $k > n$ not considered in part **(b)**?

2. See **Lipschutz** [1968], page 31, for another proof by induction.
3. Actually, the reverse of Theorem 1.1 is also true. This is given in the next corollary.

Problem Using Theorem 1.1, show that a consistent system of linear equations has infinite number of solutions if, and only if, it does not have a unique solution.

Corollary 1.1 Given a system of linear equations whose echelon form has k nonzero equations as shown in equation (1.1), we have

a. If the system of linear equations is inconsistent, then in the echelon form $b'_r \neq 0$, for some $k+1 \leq r \leq m$.
b. If the system of linear equations is consistent, then in the echelon form $b'_r = 0$, for all $k+1 \leq r \leq m$. Furthermore, if the solution is unique then $k = n$, and if there are infinite number of solutions then $k < n$.

Proof (a) By the contrapositive statement of part (b) of Theorem 1.1, if the system of linear equation is inconsistent, then in the echelon form $b'_r \neq 0$, for some $k+1 \leq r \leq m$.

(b) By the contrapositive statement of part (a) of Theorem 1.1, if the system of linear equation is consistent, then in the echelon form $b'_r = 0$, for all $k+1 \leq r \leq m$. If the system has infinite number of solutions, it does not have a unique solution, and, by the contrapositive statement of part (b.*i*), we have $k < n$. If the system has a unique solution, it does not have infinite number of solution, and, by the contrapositive statement of part (b.*ii*), we have $k = n$.❑

Problem Lipschutz [1968], page 27, #2.7. What conditions must be placed on a, b and c so that the following system in unknowns $x,\ y$ and z has a solution?

$$
\begin{aligned}
x + 2y - 3z &= a \\
2x + 6y - 11y &= b \\
x - 2y + 7z &= c
\end{aligned}
$$

Definition 1.5 A system of linear equations is ***homogeneous*** if right-hand side coefficients $b_i = 0$, for all $i = 1,2, \dots, m$, otherwise, it is ***nonhomogeneous***.

Example Markov Model of Employment (See **Simon & Blume** [1994], page 113).

Problem Show that a homogeneous system of linear equations is always consistent and, if the solutions is unique, all the variables must be zeroes. Is it true that, if there is a solution that all the variables are zeroes, then that solution is unique?

Problem **Lipschutz** [1968], page 29, #2.10. Suppose in a homogeneous system of linear equations the coefficients of one of the unknowns are all zero. Show that the system has a nonzero solution.

Problem Given the system of linear equations:

$$
\begin{aligned}
Ax + 2y + 3z &= 4 \\
-y + Bz &= D \\
Cz &= E
\end{aligned}
$$

what are the conditions on the values of A, B, C, D and E so that the system has (a) infinite number of solutions, (b) no solution, and (c) unique solution.

Problem **Fraleigh & Beauregard**, [1995], page 69, #29 a-c. Mark each of the following True or False.

_______**a)** Every linear system with the same number of equations as unknowns has a unique solution.

_______**b)** Every linear system with the same number of equations as unknowns has at least one solution.

_______**c)** A linear system with more equations than unknowns may have an infinite number of solutions.