

# EE320 Chapter 6

## Optimization without Constraints: One Independent Variable Case

Panit Wattanakoon  
Faculty of Economics, Thammasat University

### 1 Introduction

Optimal value in economics is derived either from

- maximizing some economic variables such as profit, utility or
- minimizing some economic variables such as cost, budget

by looking at the objective function (dependent variable)

and making choice (independent variable)

Example: Firm's problem:

$$\Pi(Q) \equiv R(Q) - C(Q)$$

where  $\Pi$  is the object of optimization/maximization.

$Q$  is the choice variable.

## 2 Maxima, Minima and Inflection point

### 2.1 Necessary Conditions

Consider this function  $y = f(x)$

$A, B$  = relative maxima  
 $D$  = relative minima  
 $C$  = absolute maxima  
 $E$  = absolute minima  
 $F$  = inflection point

To test that point, find 1<sup>st</sup> - order derivative

$$\left. \begin{array}{l} \frac{dy}{dx}\Big|_A = 0 \\ \frac{dy}{dx}\Big|_B = 0 \end{array} \right\} \text{relative max}$$
$$\left. \begin{array}{l} \frac{dy}{dx}\Big|_D = 0 \\ \frac{dy}{dx}\Big|_E = 0 \end{array} \right\} \text{relative min}$$
$$\left. \frac{dy}{dx}\Big|_F = 0 \right\} \text{inflection point}$$

These are “necessary condition” for relative maximum/minimum. We still cannot indicate whether it is maximum, minimum or inflection point)

Example:  $y = x^3 - 12x^2 + 36x + 8$

First-order condition:  $\frac{dy}{dx} = 3x^2 - 24x + 36 = 0$

Find tangent line that slope is zero  $\Rightarrow (x - 2)(x - 6) = 0$

## 2.2 Sufficient Condition ( $2^{nd}$ order derivative)

Suppose  $y = f(x)$

$$\frac{dy}{dx} = y' = f'(x) \quad 1^{st} \text{ order derivative}$$

$$\frac{d^2y}{dx^2} = y'' = f''(x) \quad 2^{nd} \text{ order derivative}$$

Note that  $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

Example:  $y = x^3$

$$\frac{dy}{dx} =$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) =$$

$$\frac{d^3y}{dx^3} =$$

For both  $\frac{dy}{dx} > 0$  and  $\frac{d^2y}{dx^2} > 0$  means the slope is positive and is increasing.

∴ Interpretation of 1<sup>st</sup> and 2<sup>nd</sup> derivative :

1<sup>st</sup> derivative  $f'(x) > 0$  the value of the function tends to .....  
 $f'(x) < 0$  the value of the function tends to .....

2<sup>nd</sup> derivative  $f''(x) > 0$  the slope of the function tends to .....  
 $f''(x) < 0$  the slope of the function tends to .....

Possible cases

1)  $f'(x_0) > 0$  and  $f''(x_0) > 0$  slope is ..... and ..... as x increases.

2)  $f'(x_0) < 0$  and  $f''(x_0) > 0$  slope is ..... and ..... as x increases.

3)  $f'(x_0) > 0$  and  $f''(x_0) < 0$  slope is ..... and ..... as x increases.

4)  $f'(x_0) < 0$  and  $f''(x_0) < 0$  slope is ..... and ..... as x increases.

Example:

Note inflection point:  $\frac{dy}{dx} = 0$  and  $\frac{d^2y}{dx^2} = 0$

### 3 Convexity and Concavity

#### Definition: Strictly convex

A function  $f(x)$  is strictly convex if we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely above the curve, except at points M and N.

#### Definition: Strictly concave

A function  $f(x)$  is strictly concave if we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely below the curve, except at points M and N.

Or assume that  $f(x)$  is continuous and twice differentiable:

If  $f''(x) > 0 \quad \forall X$  , then  $f(x)$  is strictly convex function.

If  $f''(x) < 0 \quad \forall X$  , then  $f(x)$  is strictly concave function.

Example: 1. Increasing convex

$$f'(x) > 0$$

$$f''(x) > 0$$

2. Decreasing convex

$$f'(x) < 0$$

$$f''(x) > 0$$

3. Increasing concave

$$f'(x) > 0$$

$$f''(x) < 0$$

4. Decreasing concave

$$f'(x) < 0$$

$$f''(x) < 0$$

Example: Attitude towards risk.

✓ Risk-averse preference  $U(x)$  is concave

✓ Risk-loving preference  $U(x)$  is convex

## Conclusion

Given a function,  $y = f(x)$ , we can identify the maximum or minimum by finding

- FOC:

$\frac{dy}{dx} = 0$  Solve for critical value that its tangent line has slope = 0

- SOC:

$\frac{d^2y}{dx^2} > 0$        $f''(x_0) > 0$  minimum (slope increases)

$\frac{d^2y}{dx^2} < 0$        $f''(x_0) < 0$  maximum (slope decreases)

Example: Determine the extreme of  $f(x) = x^3 - 3x^2 + 2$

## Chapter 6

---

Example:  $g(x) = \frac{x^4}{4} - \frac{3}{2}x^2$

Example:  $f(x) = x^4$

Example:  $f(x) = x^3$

Therefore, when we get  $f''(x_0) = 0$ , it could be

1. Maximum or Minimum ex.  $y = x^4$  slope changes its sign.
2. Inflection point ex.  $y = x^3$  slope remains in the same sign.

Conditions	Maximum	Minimum
1. First-order necessary	$f'(x) = 0$	$f'(x) = 0$
2. Second-order necessary	$f''(x) \leq 0$	$f''(x) \geq 0$
3. Second-order sufficient	$f''(x) < 0$	$f''(x) > 0$

## 4 Profit maximization problem

Consider a profit function:  $\Pi(Q) = TR(Q) - TC(Q)$ . A firm can optimize its objective function by choosing the amount of quantity produced so that it maximizes the profit:

$$\max_Q \Pi(Q) = TR(Q) - TC(Q)$$

FOC: Set  $\frac{d\Pi(Q)}{dQ} = 0 \Rightarrow$

$$\frac{d}{dQ}\Pi(Q) = 0$$

$$\frac{d}{dQ}TR(Q) - \frac{d}{dQ}TC(Q) = 0$$

$$MR(Q) - MC(Q) = 0$$

$$MR(Q) = MC(Q) \text{ profit-maximizing necessary condition}$$

For max profit, we want  $\frac{d^2\Pi(Q)}{dQ^2} < 0$

SOC:  $\frac{d^2\Pi(Q)}{dQ^2} < 0 \Rightarrow$

$$\frac{d^2\Pi(Q)}{dQ^2} = \frac{d^2TR(Q)}{dQ^2} - \frac{d^2TC(Q)}{dQ^2} < 0$$

$$MR'(Q) - MC'(Q) < 0$$

$$MR'(Q) < MC'(Q) \text{ sufficient condition}$$

$$\text{slope of MR} < \text{slope of MC}$$

Two cases to consider:

1. Perfect Competition
2. Monopoly

## 4.1 Perfect Competition

Assumptions:

1. many buyers and sellers  $\Rightarrow$  firms as price taker
2. homogeneous product
3. no barrier to entry or exit etc.

$$TR(Q) = \bar{P} \cdot Q$$

$$TC(Q) = C(Q)$$

$$\Rightarrow \Pi(Q) = TR(Q) - TC(Q)$$

FOC:

$$\frac{d}{dQ}\Pi(Q) = \bar{P} - MC(Q) = 0$$

$$\bar{P} = MC(Q)$$

SOC:

$$\frac{d^2}{dQ^2}\Pi(Q) < 0$$

$$0 - MC'(Q) < 0$$

$$MC'(Q) > 0$$

## Chapter 6

---

Example: Given the following information,  $P = 10$  ,  $TC = Q^2$ . Find  $\Pi_{max}$ ,  $Q$  and show that  $\Pi$  is maximum.

Example:  $P = 30$  ,  $TC = 100 + 19Q - 5Q^2 + \frac{1}{3}Q^3$ . Find  $Q^*$  maximizing profit

## 4.2 Monopoly

Producer/seller can set the price in the market.

$$TR(Q) = P(Q) \cdot Q$$

$$TC(Q) = C(Q)$$

$$\Rightarrow \Pi(Q) = TC(Q) - C(Q)$$

FOC:

$$\begin{aligned} \frac{d\Pi}{dQ} &= [P(Q) \cdot (1) + Q \cdot P'(Q)] - MC(Q) = 0 \\ MR(Q) &= MC(Q) \end{aligned}$$

SOC:

$$\frac{d^2}{dQ^2} \Pi(Q) < 0 \Rightarrow MR'(Q) < MC'(Q)$$

Example:  $P = 4000 - 33Q$ ,  $TC = 2Q^3 - 3Q^2 + 400Q - 5000$ , find  $Q_{max}^*$  that maximize  $\Pi$ .

Example:  $P = 48 - 0.5Q$ ,  $TC = 2 + 60Q - 8Q^2 + Q^3$ . Find  $Q^*$

$$\Pi(Q) = \underbrace{(48 - 0.5Q)Q}_{TR} - \underbrace{2 - 60Q + 8Q^2 - Q^3}_{TC}$$

### Monopoly power

$$TR = P(Q) \cdot Q$$

$$\frac{dTR}{dQ} = P(Q) \frac{dQ}{dQ} + Q \frac{dP(Q)}{dQ}$$

$$MR = P(Q) + Q \frac{d}{dQ} P(Q) \cdot \frac{P(Q)}{P(Q)}$$

$$= P(Q) + \left[ \frac{Q}{P(Q)} \cdot \frac{d}{dQ} P(Q) \right] P(Q)$$

$$= P(Q) + \frac{1}{\varepsilon_d} \cdot P(Q)$$

$$\therefore MR = P(Q) \left[ 1 + \frac{1}{\varepsilon_d} \right] \quad \text{or} \quad MR = P(Q) \left[ 1 - \frac{1}{|\varepsilon_d|} \right]$$

At profit-maximizing condition  $MR(Q) = MC(Q)$

$$P(Q) \cdot \left[1 - \frac{1}{|\varepsilon_d|}\right] = MC(Q)$$

$$1 - \frac{1}{|\varepsilon_d|} = \frac{MC(Q)}{P(Q)}$$

$\vdots$

$$\frac{MC(Q) - P(Q)}{P(Q)} = \frac{-1}{|\varepsilon_d|}$$

$$\Leftrightarrow \frac{P(Q) - MC(Q)}{P(Q)} = 1 - \frac{1}{|\varepsilon_d|}$$

$\Rightarrow$  Lerner's index

$$\frac{P(Q) - MC(Q)}{P(Q)} = 1 - \frac{1}{|\varepsilon_d|}$$

In PC,  $|\varepsilon_d| = \infty$  because producers are price taker

$$\therefore \frac{P(Q) - MC(Q)}{P(Q)} = \frac{1}{\infty} = 0 \Rightarrow P = MC(Q)$$

In monopoly,  $|\varepsilon_d| < \infty \Rightarrow \frac{1}{|\varepsilon_d|} > 0$

$$\therefore \frac{P(Q) - MC(Q)}{P(Q)} > 0 \Rightarrow P(Q) > MC(Q)$$

## 5 Effect of taxes

Consider again the profit function of a firm:  $\Pi(Q) = TR(Q) - TC(Q)$ , what will happen to  $Q^*$  if government imposes

1. lump-sum tax:  $T = t_0$  fixed amount
2. profit tax :  $T = t\Pi$  where  $0 < t < 1$
3. specific tax :  $T = tQ$  where  $0 < t < 1$

✓ lump-sum tax

$$\Pi(Q) =$$

✓ profit tax

$$\Pi(Q) =$$

✓ specific tax

$$\Pi(Q) =$$

## 6 Maximization of tax revenue

Consider a firm facing a demand function  $P = a - bQ$  which means that its total revenue is  $TR = aQ - bQ^2$ , given  $TC = c_0 + c_1Q + c_2Q^2$

Suppose government imposes tax \$t per unit;

TC after tax:  $TC_T =$

$\Pi$  after tax:  $\Pi_T =$

Profit-maximizing condition:

FOC:

$$\frac{d}{dQ} \Pi_T =$$

=

$$Q^* =$$

SOC:

$$\frac{d^2 \Pi_T}{dQ^2} =$$

$\therefore$  Total tax revenue:  $t \cdot Q^* =$

Tax-revenue maximizing condition:

$$\max_t T(t) = \frac{(a - c_1 - t)t}{2(b + c_2)}$$

FONC:  $\frac{d}{dt}T(t) =$

SOSC:  $\frac{d^2}{dt^2}T(t) =$

Example:  $P = 40 - 0.5Q$ ,  $TC = 2 - 5Q + 7Q^2$ . Find  $t$  (specific tax) that maximizes total tax revenue.

## Reference

Aroonruengsawat, Anin. EE320 Lecture Handouts.

Chiang, A. C. and Wainwright, K. (2005) “Fundamental Methods of Mathematical Economics,” 4th edition, McGraw-Hill, Inc., Singapore.

Dowling, E. T. (2001) “Schaums Outline of Theory and Problem of Introduction to Mathematical Economics”, 3rd edition, The McGraw-Hill Companies, Inc.

Holden, K. and Pearson, A.W. (1992) “Introductory Mathematics for Economics and Business,” Second edition, The Macmillan Press Ltd.

Sydsaeter, K. and P. Hammond. (2006) “Essential Mathematics for Economic Analysis,” 2nd edition, Prentice Hall.