Assignment 4

DUE DATE: Tuesday 9nd, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

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Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$R_t - \phi_1 R_{t-1} + \phi_2 R_{t-2} = \phi_0 + a_t$$

$$(1 - 1.5B + 0.9B^2)R_t = 0.25 + \varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean (μ) = 0 and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

① the reversed characteristic equation :

$$B^2 - 1.5B + 0.9 = 0$$

$$\lambda_i = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

$$\lambda_i = \frac{-(-1.5) \pm \sqrt{(-1.5)^2 - 4(0.9)(1)}}{2(1)}$$

$\lambda_1 = 0.75 + \frac{3\sqrt{15}}{20} i \rightarrow$ modulus = 0.9487

$\lambda_2 = 0.75 - \frac{3\sqrt{15}}{20} i \rightarrow$ modulus = 0.9487

Since the module of the reversed characteristic eqn is 0.9487 which less than 1. Therefore, R_t is weak stationarity.

Question 1.2 (10 points)

Your score......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

conditional mean : $E(R_t|F_{t-1})$

$R_t = \phi_0 + \phi_1 R_{t-1} + \phi_2 R_{t-2} + a_t$

Multiply $E(\cdot|F_{t-1})$ on both side

$E[R_t|F_{t-1}] = E[\phi_0|F_{t-1}] + \phi_1 E[R_{t-1}|F_{t-1}] + \phi_2 E[R_{t-2}|F_{t-1}] + E[\cancelto{0}{a_t}|F_{t-1}]$

$E[R_t|F_{t-1}] = \phi_0 + \phi_1 R_{t-1} + \phi_2 R_{t-2}$

$E[R_t|F_{t-1}] = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2}$

unconditional mean : $E(R_t)$

$R_t = \phi_0 + \phi_1 R_{t-1} + \phi_2 R_{t-2} + a_t$

Multiply $E(\cdot)$ to both side

$E(R_t) = E(\phi_0) + \phi_1 E(R_{t-1}) + \phi_2 E(R_{t-2}) + E(\cancelto{0}{a_t})$

$E(R_t) = \phi_0 + \phi_1 E(R_{t-1}) + \phi_2 E(R_{t-2})$

Since R_t is weak stationarity. Then, $E(R_t) = E(R_{t-1}) = E(R_{t-2})$

$(1 - \phi_1 - \phi_2) R_t = \phi_0$

$$R_t = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

$$R_t = \frac{0.25}{1 - 1.5 + 0.9} = 0.625$$

Question 1.3 (10 points) | Your score......................

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

Unconditional variance : $var(R_t)$

$$(R_t - \mu) = \phi_1 (R_{t-1} - \mu) + \phi_2 (R_{t-2} - \mu) + a_t$$

square and Multiply $E(\cdot)$ to both side, respectively

$$E[(R_t-\mu)^2] = \phi_1^2 \overbrace{E[(R_{t-1}-\mu)^2]}^{var(R_{t-1})} + \phi_2^2 \overbrace{E[(R_{t-2}-\mu)^2]}^{var(R_{t-2})} + \overbrace{E[a_t^2]}^{\sigma^2} + 2\phi_1\phi_2 \overbrace{E[(r_{t-1}-\mu)(r_{t-2}-\mu)]}^{\gamma_1}$$

$$= 2\phi_1 \overbrace{E[(R_{t-1}-\mu)(a_t)]}^{0} + 2\phi_2 \overbrace{E[(R_{t-2}-\mu)(a_t)]}^{0}$$

$$var(R_t) = \phi_1^2 var(R_{t-1}) + \phi_2^2 var(R_{t-2}) + \sigma^2 + 2\phi_1\phi_2\gamma_1$$

since R_t is weak stationarity. Then, $var(R_t) = var(R_{t-1}) = var(R_{t-2})$

$$var(R_t) = \frac{\sigma^2 + 2\phi_1\phi_2\gamma_1}{1-\phi_1^2-\phi_2^2} = \frac{(0.25) + 2(1.5)(-0.9)(0.8\,var(R_t))}{1-(1.5)^2-(-0.9)^2} = 3.493$$

$$(R_t-\mu) = \phi_1(R_{t-1}-\mu) + \phi_2(R_{t-2}-\mu) + a_t$$

Multiply $(R_{t-j}-\mu)$ to both side + take $E[\cdot]$

$$E[(R_t-\mu)(R_{t-j}-\mu)] = \phi_1 E[(R_{t-1}-\mu)(R_{t-j}-\mu)] + \phi_2 E[(r_{t-2}-\mu)(r_{t-j}-\mu)] + E[a_t(r_{t-j}-\mu)]$$

$$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2}$$

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_{-1} \quad ; \ \gamma_{-1} = \gamma_1 \text{ from weak stationarity}$$

$$\gamma_1 = 1.5\gamma_0 - 0.9\gamma_1$$

$$\gamma_1 = \frac{1.5}{1.9} var(r_t)$$

Conditional variance : $Var(R_t|F_{t-1})$

$$R_t = \phi_0 + \phi_1 R_{t-1} + \phi_2 R_{t-2} + a_t$$

multiply $var(\cdot|F_{t-1})$ to both side

$$var(R_t|F_{t-1}) = var(\phi_0|F_{t-1}) + \phi_1^2 var(R_{t-1}|F_{t-1}) + \phi_2^2 var(R_{t-2}|F_{t-1}) + Var(a_t) + 2cov(\phi_1 R_{t-1}, \phi_2 R_{t-2}) + 2cov(\phi_1 R_{t-1}, a_t) + 2cov(\phi_2 R_{t-2}, a_t)$$

$$= 0 + \phi_1^2(0) + \phi_2^2(0) + \sigma^2 + 0 + 0$$

$$var(R_t|F_{t-1}) = \sigma^2$$

$$= 0.25$$

Question 1.4 (10 points) Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

$(R_t - \mu) = \phi_1 (R_{t-1} - \mu) + \phi_2 (R_{t-2} - \mu) + a_t$

Multiply $(R_{t-j} - \mu)$ to both side + take $E[\cdot]$

$$E[(R_t-\mu)(R_{t-j}-\mu)] = \phi_1 E[(R_{t-1}-\mu)(R_{t-j}-\mu)] + \phi_2 E[(r_{t-2}-\mu)(r_{t-j}-\mu)] + E[a_t(r_{t-j}-\mu)]$$

$$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} = 1.5\gamma_{j-1} - 0.9\gamma_{j-2}$$

divide by γ_0

$$\frac{\gamma_j}{\gamma_0} = \phi_1 \frac{\gamma_{j-1}}{\gamma_0} + \phi_2 \frac{\gamma_{j-2}}{\gamma_0}$$

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}$$

① $j=1$;

$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_{-1}$; $\rho_{-1} = \rho_1$ from stationarity and $\rho_0 = 1$

$\rho_1 = 1.5 - 0.9\rho_1$

$\rho_1 = \frac{1.5}{1.9} = \frac{15}{19}$

and $j=2$;

$\rho_2 = 1.5\rho_1 - 0.9\rho_0 = 1.5\left(\frac{15}{19}\right) - 0.9 = \frac{27}{95}$

Question 1.5 (10 points) Your score......................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

1-step forecast: $r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \ldots + \phi_p r_{t-p} + a_t$

$r_{n+1} = \phi_0 + \phi_1 r_n + \phi_2 r_{n-1} + \ldots + \phi_p r_{n+1-p} + a_t$ and $F_n = r_n, r_{n-1} \ldots$

$\hat{r}_n(1) = E[r_{n+1} | F_n]$

$= E[\phi_0 | \cdot] + E[\phi_1 r_n | \cdot] + \ldots + E[\phi_p r_{n+1-p} | \cdot] + E[\cancel{a_{n+1}} | \cdot]$ 0

$= \phi_0 + \phi_1 r_n + \phi_2 r_{n-1} + \ldots + \phi_p r_{n+1-p}$

$= 0.25 + 1.5(0.01) + (-0.9)(0.02) = 0.247$

$r_{n+1} - \hat{r}_n(1) = a_{n+1} | e_n(1)$

$var(e_n(1) | \cdot) = \cancel{var(a_{n+1} | \cdot)}$ σ_a^2

estimation (interval) $= \hat{r}_n(1) \pm Z_{\frac{\alpha}{2}} \sqrt{var(a_{n+1} | \cdot)} = 0.247 \pm 1.96(0.5)$

$-0.733 \leq R_{1001} \leq 1.227$ at 95% CI #

2-step forecast: $E[r_{n+2} | \cdot] = \hat{r}_n(2)$

$= \phi_0 + \phi_1 \hat{r}_n(1) + \phi_2 r_{n-1}$

$= 0.25 + 1.5(0.247) + (-0.9)(0.02) = 0.6025$

$r_{n+2} - \hat{r}_n(2) = a_{n+2} | e_n(2) = \phi_1 [\underbrace{r_{n+1} - \hat{r}_n(1)}_{a_{n+1}}] + a_{n+2}$

$= \phi_1 a_{n+1} + a_{n+2}$

$var(e_n(2) | \cdot) = \phi_1^2 var(a_{n+1} | \cdot) + var(a_{n+2} | \cdot) + 2cov[\phi_1 a_{n+1}, a_{n+2} | \cdot]$

$= \phi_1^2 \sigma_a^2 + \sigma_a^2 \rightarrow (1 + \phi_1^2) \sigma_a^2$

$= (1 + 1.5^2) \sigma_a^2 = 3.25 \sigma_a^2$

estimation (interval) $= \hat{r}_n(2) + Z_{\frac{\alpha}{2}} \sqrt{var(a_{n+2} | \cdot)}$

$= 0.6025 \pm 1.96 \sqrt{3.25(0.25)}$

$= 0.6025 \pm 1.767$

$1.52 \leq \hat{r}_{1000}(2) \leq 2.014$ at 95% CI #