FN211: Lecture Note 2-3
Random Variables and Their Distributions

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Background: What are investment returns?

- Investment returns measure financial results of an investment.
- Returns may be historical or prospective (anticipated).
- Returns can be expressed in:
  - ($) dollar terms.
  - (%) percentage terms.

Definition

(1) Discrete Compounding

- If the interest rate is simple (no compounding takes place), then the future value of any investment can be written as:

$$\text{Future Value (FV)} = PV \times (1 + r)^t$$

How to compute return?

Definition

(2) Continuous Compounding

- Continuous compounding introduces the concept of the natural logarithm. This is the constant rate of growth for all naturally growing processes. It's a figure that developed out of physics.

$$FV = P * e^{rt}$$

How to compute return?

What is difference?

Period	0	1	2
Asset Value	100	90	100

What are returns and total returns?

Note to VaR Derivation

- VaR is often defined in dollars, denoted by \$VaR
- \$VaR loss is implicitly defined from the probability of getting an even larger loss as in

$$\Pr(\$Loss > \$VaR) = p$$

- Note by definition that (1–p)100% of the time, the \$Loss will be smaller than the VaR.
- Define VaR with the unit of percentage:

$$\Pr(-R_{PF} > VaR) = p \Leftrightarrow$$
$$\Pr(R_{PF} < -VaR) = p$$

Note to VaR Derivation

Outline

How to utilize asset return for financial analysis?

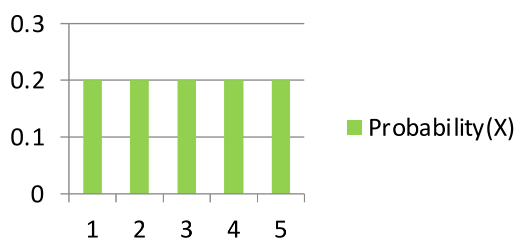
- Discrete Random Variables and Their Distribution
- Continuous Random Variables and Their Distribution
- Application in Finance

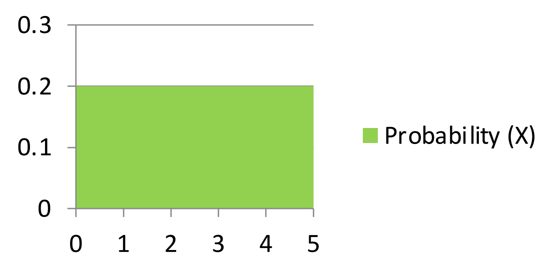
Discrete Random Variables and Their Distribution

Discrete and Continuous Random Variables

A random variable is a variable whose future values are uncertain.

- **Discrete random variables** have a theoretically countable number of outcomes.
  - There may be an infinite number of them, but they are countable.
  - Price is a discrete random variable.
- **Continuous random** variables have a theoretically uncountable number of outcomes.
  - Rate of return is a continuous random variable.
  - Temperature is a continuous random variable.





Probability distribution

The set of probabilities for the possible outcomes of a random variable is called a "probability distribution."

- The underlying foundation of most inferential statistical analysis is the concept of a probability distribution.
- The focus in the investment arena is on four probability distributions.
    1. Uniform
    2. Binomial
    3. Normal
- An understanding of probability distributions is critical to using further quantitative methods such as hypothesis testing, regression, and time-series analysis.

The probability density function (pdf)

The mathematical expression that describes the individual probabilities that a random variable will take on each of a set of specified values is known as its probability density function.

- For **a discrete distribution**, the pdf has discrete, countable, nonzero probabilities for every possible outcome.
- For **a continuous distribution**, the pdf has continuous, uncountable probabilities for each possible specified outcome in the set of infinite, uncountable outcomes.
  - For continuous distributions, this result means that the cumulative distribution function will be more useful and, to some extent, more meaningful.

$$\text{pdf}(\text{Div}) = \begin{cases} P(\text{Div} = \$1) = 0.05 \\ P(\text{Div} = \$5) = 0.30 \\ P(\text{Div} = \$7) = 0.50 \\ P(\text{Div} = \$10) = 0.10 \\ P(\text{Div} = \$11) = 0.05 \end{cases}$$

The cumulative distribution function (cdf)

The mathematical expression that describes the probability that a random variable will be less than or equal to a specific value for all possible values of that variable is known as its cumulative distribution function.

- The cumulative distribution function, denoted *F(x)*, is represented as the sum of the probabilities of the specified outcome and all prior outcomes in the distribution for each and every possible outcome.
  - By analogy, this is very similar to the concept of cumulative relative frequency from Chapter 3 on Statistical Concepts and Market Returns.
- The cdf has the same properties as the pdf, in addition to
  1. All values of the cdf are between 0 and 1;
  2. As we increase the value of the specified outcome, the cdf must increase or remain constant.

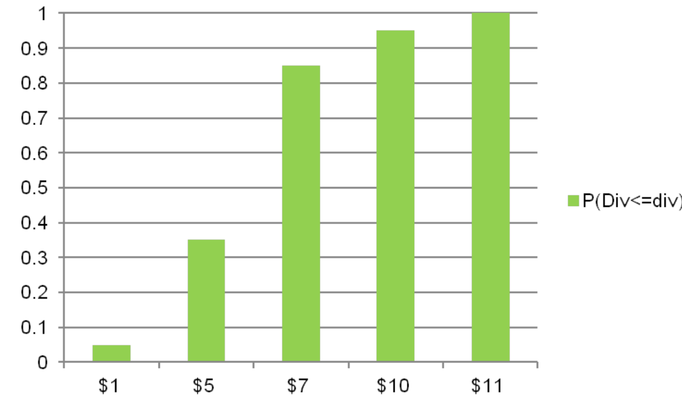
$$\text{pdf(Div)} = \begin{cases} P(\text{Div} = \$1) = 0.05 \\ P(\text{Div} = \$5) = 0.30 \\ P(\text{Div} = \$7) = 0.50 \\ P(\text{Div} = \$10) = 0.10 \\ P(\text{Div} = \$11) = 0.05 \end{cases}$$

$$\text{cdf(Div)} = \begin{cases} P(\text{Div} \leq \$1) = 0.05 \\ P(\text{Div} \leq \$5) = 0.35 \\ P(\text{Div} \leq \$7) = 0.85 \\ P(\text{Div} \leq \$10) = 0.95 \\ P(\text{Div} \leq \$11) = 1.00 \end{cases}$$

The cdf in action

- **Example:** Returning to the special dividend example, the cdf can be written and depicted as:

$$\text{cdf(Div)} = \begin{cases} P(\text{Div} \le \$1) = 0.05 \\ P(\text{Div} \le \$5) = 0.35 \\ P(\text{Div} \le \$7) = 0.85 \\ P(\text{Div} \le \$10) = 0.95 \\ P(\text{Div} \le \$11) = 1.00 \end{cases}$$



- What is the probability of receiving at most a $10 dividend?

- What is the probability of receiving more than a $7 dividend?

PDF and CDF

Definition: Let F(*y*) be the distribution function for a continuous random variable Y. Then *f*(*y*), given by:

$$f(y) = \frac{dF(y)}{dy} = F'(y)$$

wherever the derivative exists, is called the ***probability density function*** (***pdf***) for the random variable Y.

Recall the Fundamental Theorem of Calculus:

$$F(y) = \int_{-\infty}^{y} \frac{dF(t)}{dt} dt$$

$$= \int_{-\infty}^{y} f(t)dt \quad = \quad P(Y \leq y)$$

The discrete UNIFORM DISTRIBUTION

A basic distribution wherein the probably of every possible countable outcome is equally likely.

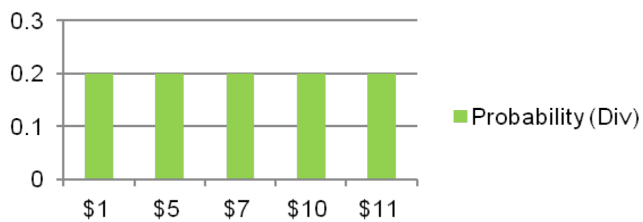
Consider again our special dividend with five possible year-end outcomes of \$1, \$5, \$7, \$10, and \$11, except that **now the probability of each outcome is 0.2, so this is a discrete uniform random variable.**

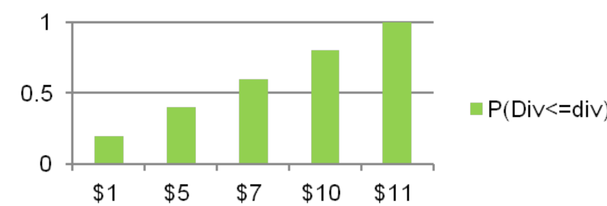
- The outcomes are countable.
- Each possible outcome has a probability between 0 and 1.
- The sum of the probabilities of the outcomes is 1.0 = 0.2 + 0.2 + 0.2 + 0.2 + 0.2.
- AND, the probability of each outcome is $1/n = 0.2$.
    - Our new special dividend is a discrete uniformly distributed random variable.

The discrete UNIFORM DISTRIBUTION

Example: What is the probability we will receive a $5 dividend? A dividend of **at least** $5?

$$\text{pdf}(\text{Div}) = \begin{cases} P(\text{Div} = \$1) = 0.20 \\ P(\text{Div} = \$5) = 0.20 \\ P(\text{Div} = \$7) = 0.20 \\ P(\text{Div} = \$10) = 0.20 \\ P(\text{Div} = \$11) = 0.20 \end{cases} \qquad \text{cdf}(\text{Div}) = \begin{cases} P(\text{Div} \le \$1) = 0.20 \\ P(\text{Div} \le \$5) = 0.40 \\ P(\text{Div} \le \$7) = 0.60 \\ P(\text{Div} \le \$10) = 0.80 \\ P(\text{Div} \le \$11) = 1.00 \end{cases}$$





- A dividend that is greater than $9?

BINOMIAL Random Variables

A binomial random variable has only two possible outcomes, termed "success" and "failure" by convention.

- The basic building block of the binomial distribution is a Bernoulli random variable.
  - A Bernoulli random variable is one for which there are only two possible outcomes, and the probability of these outcomes satisfies the conditions for a valid pdf. That is, each probability is between 0 and 1 and they sum to 1.
  - A single observation of the outcome of a Bernoulli random variable is called a "trial" when the random variable can repeat.
- The sum of a series of Bernoulli trials is distributed as a binomial random variable.
- In order to use the binomial distribution, we must satisfy two conditions:

1. The probability of each outcome must be constant for all trials; and
2. The trials must be independent.

BINOMIAL random variables

Characterizing the Distribution

- The pdf for a discrete binomial distribution is written as:

$$p(x) = P(X = x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

- We indicate that a random variable is binomially distributed as: $X \sim B(n, p)$
- The mean and variance of a binomially distributed variable

	Mean	Variance
Bernoulli, $B(1,p)$	p	$p(1 - p)$
Binomial, $B(n,p)$	np	$np(1 - p)$

BINOMIAL Random Variables

Example:

- You decide to assess an analyst's ability to forecast the sufficiency of earnings over a 20-quarter period. Over that time, the analyst correctly predicted earnings 13 times and incorrectly predicted earnings 7 times. You decide to model the "correctness" of his predictions using the binomial distribution.
  - What is your estimate of the probability of a successful prediction by this analyst?
- Assuming the estimated probability is the actual probability, answer the following questions:
  - What is the probability that the analyst will be correct for the next four quarters?
  - What is the expected number of quarters the analyst will be correct over the next three years?
  - What is the standard deviation of "correctness" for that period?

Example: Expected Number of Defaults Example

- Determine the number of expected defaults in a bond portfolio with 25 issues.
  - The estimated annual default rate is .107.

1. Over the next year, what is the expected number of defaults in the portfolio, assuming a binomial model for defaults?

2. Estimate the standard deviation of the number of defaults over the coming year.

3. Critique the use of the binomial probability model in this context.

Continuous Random Variables and Their Distribution

Note: The continuous uniform distribution

Characterizing the Distribution

- The pdf and cdf for a continuous uniform distribution are written as

$$f(x) = \begin{cases} \frac{1}{b-a} \text{ when } a < x < b \\ 0 \text{ otherwise} \end{cases} \qquad F(x) = \begin{cases} 0 \text{ when } x \le a \\ \frac{x-a}{b-a} \text{ when } a < x < b \\ 1 \text{ when } x \ge b \end{cases}$$

- Probabilities are calculated from $P(a \le x \le b) = \int_a^b f(x)dx$

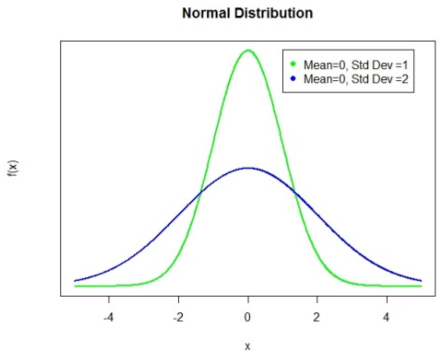
- The mean and variance of a continuous uniformly distributed variable are

$$\mu = \frac{a+b}{2} \qquad \sigma^2 = \frac{(b-a)^2}{12}$$

The Normal distribution

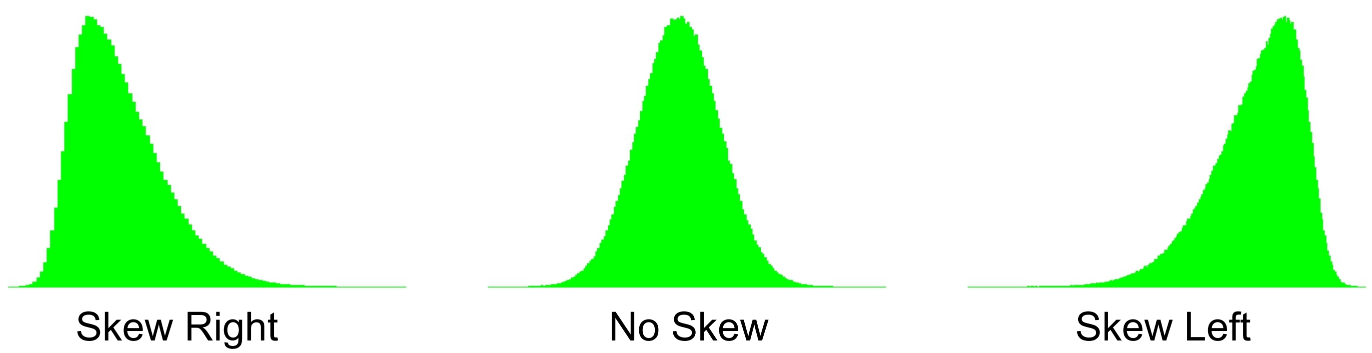
A continuous, symmetrical distribution that is completely described by its mean and variance.

- Mean, median, and mode are equal.
- The normal distribution has skewness of zero.
  - Option returns are skewed; hence, they are not normally distributed.
- Kurtosis of 3 or excess kurtosis of 0 (3 – 3 = 0).
  - k > 3 → fat tails → underestimated probability of extreme values (the blue distribution has excess kurtosis).
  - This area is one for which the normal distribution is a poor approximation for stock returns, which have "fat tails."



Skewness: $S = \frac{E(X-\mu)^3}{\sigma^3}$

- The degree of symmetry in the dispersion of values around the mean is known as skewness.
- If observations are equally dispersed around the mean, the distribution is said to be symmetrical.
- If the distribution has a long tail on one side and a "fatter" distribution on the other side, it is said to be skewed in the direction of the long tail.



Kurtosis: $K = \frac{E(X-\mu)^4}{[E(X-\mu)^2]^2}$

- Kurtosis measures the relative amount of "peakedness" as compared with the normal distribution, which has a kurtosis of 3.
  - We typically express this measure in terms of excess kurtosis being the observed kurtosis minus 3.
  - Distributions are referred to as being

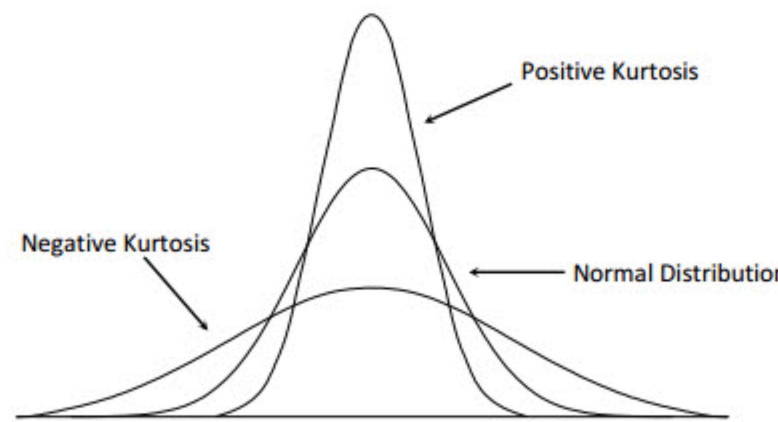
1. Leptokurtic

   (more peaked than the normal; fatter tails)

2. Platykurtic

   (less peaked than the normal; thinner tails) or

3. Mesokurtic (equivalent to the normal).



The Normal distribution

Characterizing the Normal Distribution

- The pdf for a normal distribution is written as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \text{ for } -\infty < x < \infty$$

- We indicate that a random variable is normally distributed as: $X \sim N(\mu, \sigma^2)$

- The mean and variance of a normally distributed variable are

$$\mu \approx \bar{x} \qquad \sigma^2 = s^2$$

Note (1) - Standard Normal

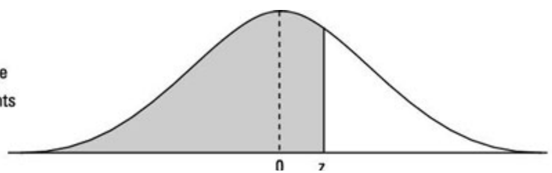
A normal distribution with a mean of 0 and standard deviation of 1 is called "standard normal."

- The prevalence of the normal distribution has led to a process whereby probability tables that have been calculated for a standard normal distribution can be used to make probability statements for any normally distributed variable.
- This process is known as "standardizing" and is accomplished by:
  1. Taking the observation(s) of interest and subtracting the mean of that observation's observed distribution;
  2. Dividing the result by the observed distribution's standard deviation.

$$Z = \frac{X - \mu}{\sigma} \qquad z = \frac{x - \bar{x}}{s}$$

Note (1) – Standard Normal

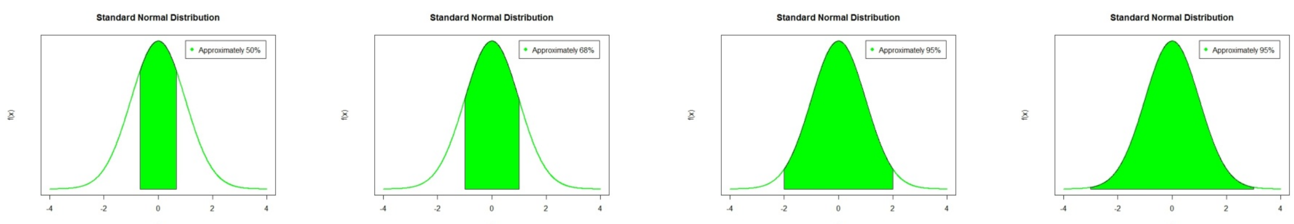
Number in the table represents P(Z ≤ z)



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

The normal distribution

- Approximately 50% of all observations fall in the interval μ ± (2/3)σ.
- Approximately 68% of all observations fall in the interval μ ± σ.
- Approximately 95% of all observations fall in the interval μ ± 2σ.
- Approximately 99% of all observations fall in the interval μ ± 3σ.



- We generally don't observe population mean and variance (μ and σ), but we can estimate them with sample mean and variance.
  - When we do, the same intervals apply, with the sample mean and variance used in place of their population analogs.

Example: Confidence Interval Calculations

- Your client's portfolio has a mean monthly return of 1.2% with a standard deviation of 3.7%. You assume for now that returns are normally distributed.
- Your client's return can be expected to fall in what range 50% of the time?
- 68% of the time?
- 95% of the time?

Example: The Standard normal distribution

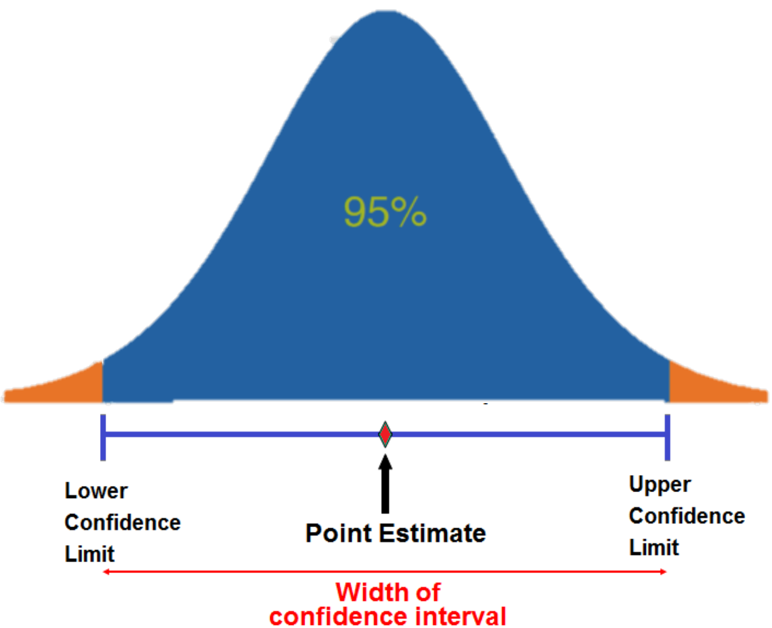
- Your client's portfolio has a mean monthly return of 1.2% with a standard deviation of 3.7%. You assume for now that returns are normally distributed.
  - What is the chance that returns will be between –2.5% and 4.9%?
  - What is the chance that returns will be negative?
- A stop-loss order automatically sells the stock if the price is below a set amount. You can set a stop-loss so that the portfolio is liquidated when it is triggered. How often will such a stop-loss be triggered if you set it so that it triggers when losses are below 1%?

Note (2) – Point and Interval Estimation

- **Point Estimate** – A sample statistic used to estimate the **exact value** of a population parameter
- **Confidence interval** (*interval estimate*) – **A range of values** defined by the confidence level within which the population parameter is **estimated** to fall.
- **Confidence Level** – The likelihood, expressed as a **percentage** or a **probability**, that a specified interval will contain the population parameter.

Note (2) – Point and Interval Estimation

Confidence Interval Width: $\bar{Y} \pm Z\left(\frac{s_Y}{\sqrt{N}}\right)$



Note:

- Increasing our confidence level from 95% to 99% means we are less willing to draw the wrong conclusion – we take a 1% risk (rather than a 5%) that the specified interval does not contain the true population mean.
- If we reduce our risk of being wrong, then we need a wider range of values

Example: Common Stocks and Normality (1)

You have a portfolio with

- Weighted average forecast mean of 0.12.
- Forecast standard deviation of 0.22.

1. Calculate and interpret a one-standard-deviation confidence interval for portfolio return, with a normality assumption for returns.
2. Calculate and interpret a 90% confidence interval for portfolio return, with a normality assumption for returns.
3. Calculate and interpret a 95% confidence interval for portfolio return, with a normality assumption for returns.

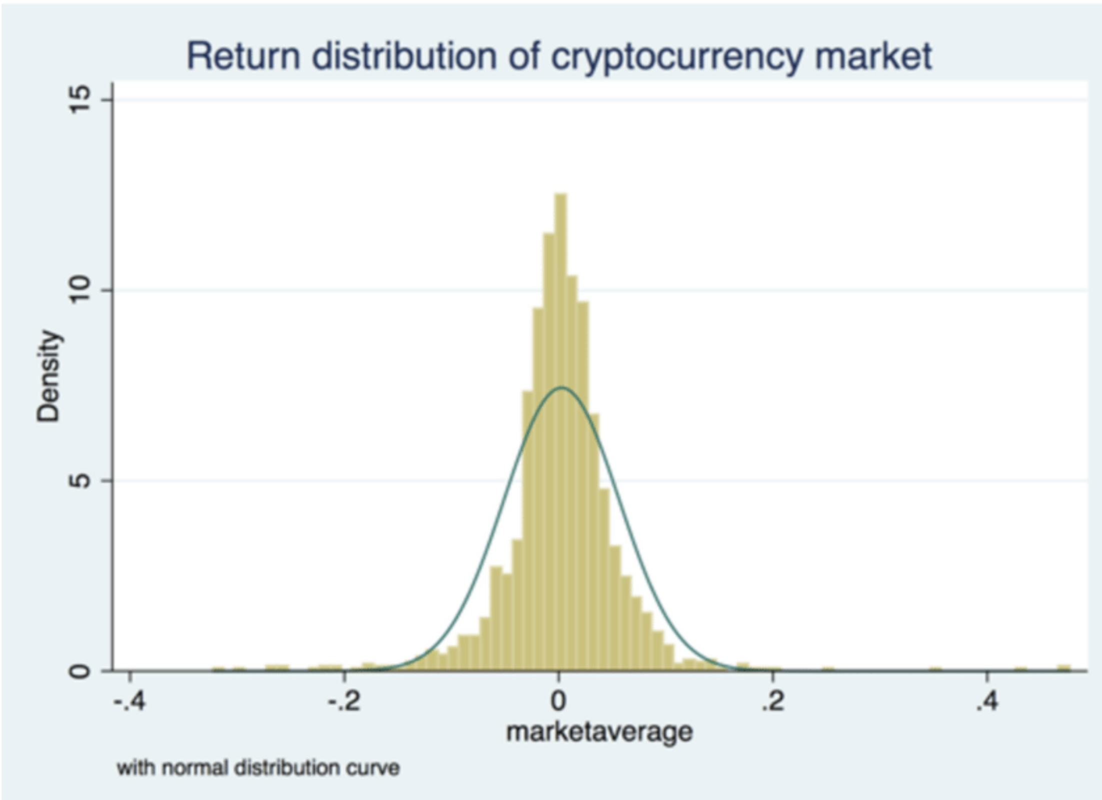
Example: Common Stocks and Normality (2)

- What is the probability that portfolio return will exceed 20%?
- What is the probability that portfolio return will be between 12% and 20%? In other words, what is P(12% ≤ Portfolio return ≤ 20%)?
- You can buy a one-year T-bill that yields 5.5%. This yield is effectively a one-year risk-free interest rate. What is the probability that your portfolio's return will be equal to or less than the risk-free rate?

Application in Finance:

Investment Returns

How to get return distribution?

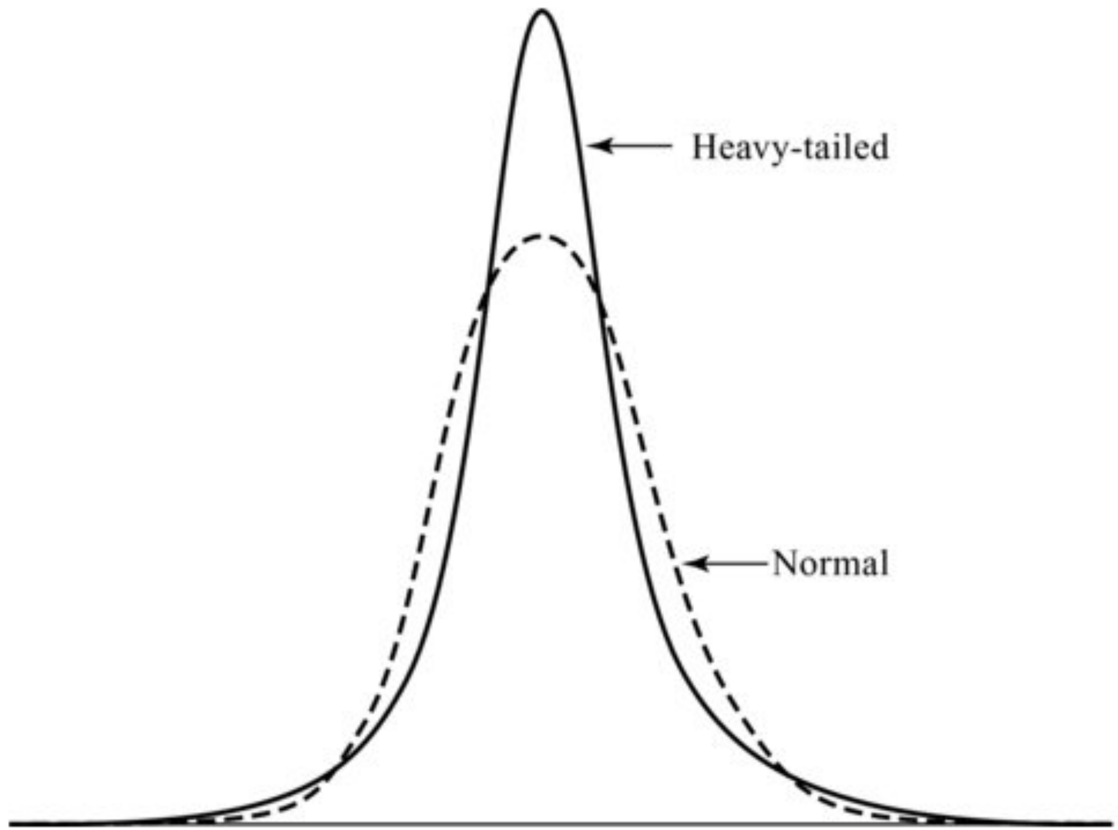


Does it follow normal distribution?

Does it follow normal distribution?

- Daily changes are **NOT normally distributed**
  - The distribution has **heavier tails** than the normal distribution
  - It is **more peaked** than the normal distribution
- This means that **small changes and large changes are more likely** than the normal distribution would suggest
- Many market variables have this property, known as **EXCESS KURTOSIS**

Heavy-Tailed vs. Normal Distribution



Alternatives to Normal Distributions: The Power Law

- An alternative to assuming normal distributions.
- It is approximately true that the value of the variable, *v*, has the property that when *x* is large:

$$Prob(v > x) = Kx^{-\alpha}$$

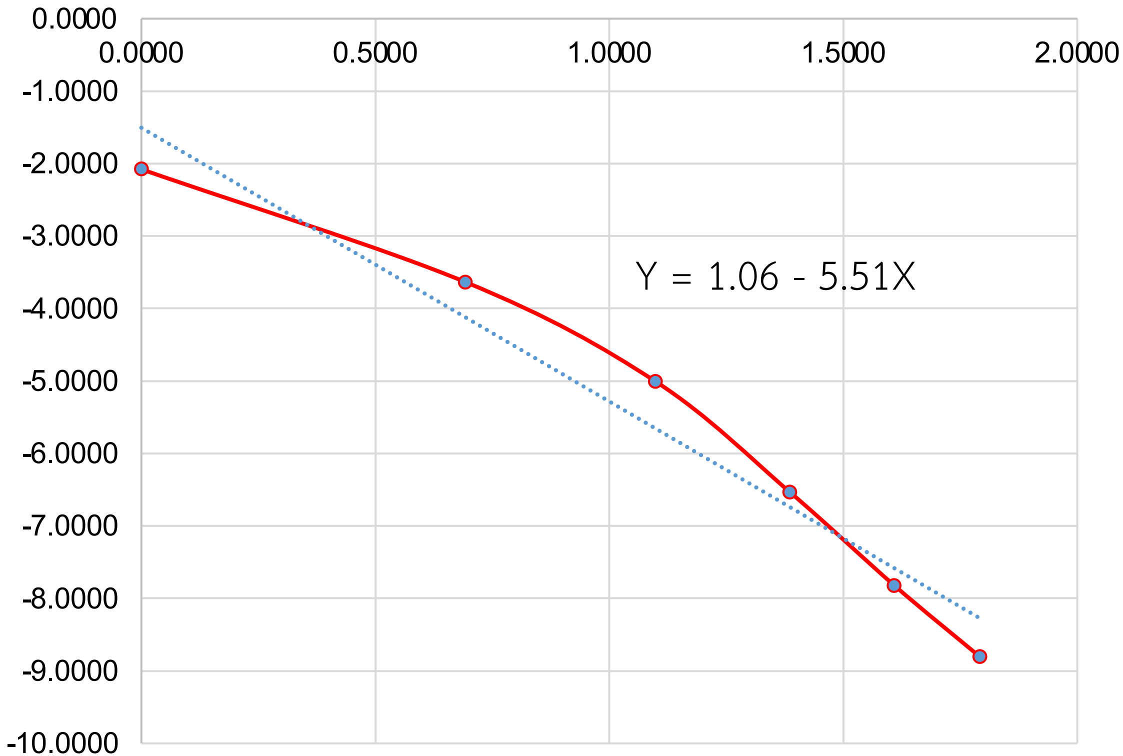
$$\ln\left[Prob(v > x)\right] = \ln K - \alpha \ln x$$

- This seems to fit the behavior of the returns on many market variables better than the normal distribution

Alternatives to Normal Distributions: The Power Law

	State	Real World (%)	Normal Model (%)	ln(x)	Prob(v>x)	ln[Prob(v>x)]
>1 SD	1	25.04	31.73	0.0000	**0.1252**	**-2.0778**
>2SD	2	5.27	4.55	0.6931	**0.02635**	**-3.6363**
>3SD	3	1.34	0.27	1.0986	**0.0067**	**-5.0056**
>4SD	4	0.29	0.01	1.3863	**0.00145**	**-6.5362**
>5SD	5	0.08	0	1.6094	**0.0004**	**-7.8240**
>6SD	6	0.03	0	1.7918	**0.00015**	**-8.8049**

Alternatives to Normal Distributions: The Power Law

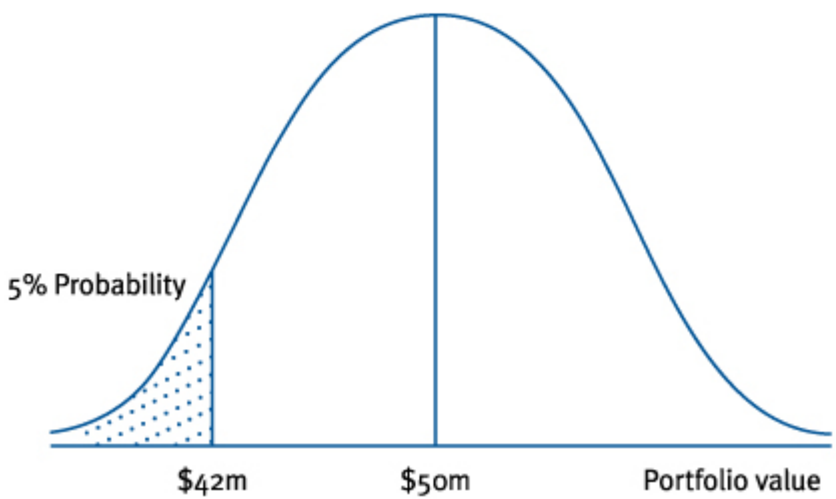


What are K and alpha?

What is VaR at 99% confidence level?

Recall: Value at Risk (VaR)

"What loss level is such that we are X% confident it will not be exceeded in N business days?"



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Question?