3. Mean-Variance Analysis

EE431: Sicha

Copeland, Thomas E. and J. Fred Weston, Financial Theory and Corporate Policy (4th ed), Addison-Wesley(2005), HG4011 .C622: Ch5 (pp 101 -141)

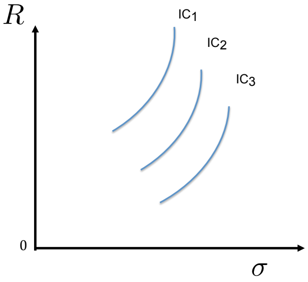
2015

Outline

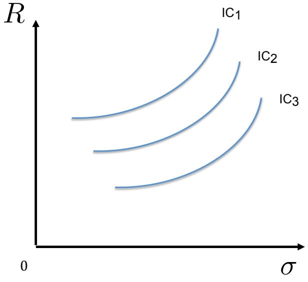
1. Measuring Risk and Returns for a Single Asset
2. Measuring Portfolio Risk and Returns
3. Efficient Frontier with Two Risky Assets
4. Efficient Frontier with One Risky and One Risk Free Asset
5. Optimal Portfolio Choice N Risky asset
6. Optimal Portfolio Choice N Risky asset and One Risk Free Asset
7. Portfolio Diversification and Individual Asset Risk

- Is it true that risk averters do not take any degree of risk?
- Risk averse individuals dislike risk but this doesn't mean they are never prepared to take any risks. It just means they need to be compensated for the risk they take by way of a greater expected return
- State preference framework = general approach (used extensively in economic theories)
- It is difficult (if not possible) to list all payoffs offered in different state of nature.
- mean-variance analysis (less general, more practical)
- Risks: deviation from expected outcome (both downside and upside, both favourable and unfavourable)
- Expected outcome = expected return
- Risks: variance, standard deviation
- require rate of return, discount rate $\rightarrow$ asset prices

- Risk $\rightarrow$ Required Rate of Return $\rightarrow$ asset prices
- What are all possible combinations between “risk” and “return” available in the financial market?



R
IC1
IC2
IC3
0
σ



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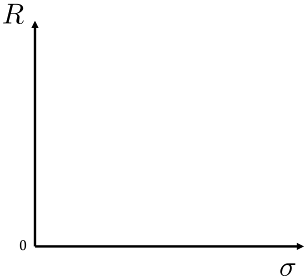


- $E(X) = \mu = \sum_{i=1}^{n} p_i X_i$
- $\sigma^2 = E[(x-\mu)^2]$, S.D = $\sigma = \sqrt{E[(x-\mu)^2]}$
- Draw indifferent curve

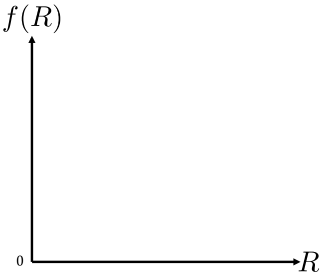
1. Measuring Risk and Returns for a Single Asset

Example 1:

p_i=probabilty	End-of-Period Price per share	R_i=Return(%)
0.1	\$20.00	
0.2	\$22.50	
0.4	\$25.00	
0.2	\$30.00	+20
0.1	\$40.00	+60
1	(current price = \$25)	

- $E(X) = \sum_{i=1}^{n} p_i X_i$
- Expected end-of-period price
  $$E(P) = 0.1 \times (......) + 0.2 \times (......) + 0.4 \times (......) + 0.2 \times (......) + 0.1 \times (......) = ...$$
- Expected Return
  $$E(R) = \frac{E(P) - P_0}{P_0} =$$
  or (percentage number)
  $$E(R) = 0.1 \times (......) + 0.2 \times (......) + 0.4 \times (......) + 0.2 \times (......) + 0.1 \times (......) = 6.........$$
  or (decimal number)
- $$E(R) = 0.1 \times (......) + 0.2 \times (......) + 0.4 \times (......) + 0.2 \times (......) + 0.1 \times (......) = 0.06$$

- Property 1: $E(X + a) = E(X) + a$
  - proof : $E(X + a) = \sum_{i=1}^{n} p_i(X_i + a)$
- Property 2: $E(aX) = aE(X)$
  - proof : $E(aX) = \sum_{i=1}^{n} p_i(aX_i)$



- Dispersion $\rightarrow$ how the returns(outcome) are distributed
- Statistically, there are many measurements (for example, range).
- Here, we focus on variance and stardard deviation.

- $VAR(X) = \sum_{i=1}^{n} p_i(X_i - E(X))^2$
- From the previous example, $E(X)$ = ...%.
- Percentage number
- $$\begin{aligned} VAR(X) &= \sum_{i=1} p_i(X_i - E(X))^2 \\ &= 0.1(..... - ......)^2 + 0.2(..... - .....)^2 + 0.4(..... - .....)^2 \\ &\quad +0.2(..... - ......)^2 + 0.1(...... - ....)^2 = ... \end{aligned}$$
- Decimal number
- $$\begin{aligned} VAR(X) &= \sum_{i=1}^{n} p_i(X_i - E(X))^2 \\ &= 0.1(..... - .....)^2 + 0.2(..... - .....)^2 + 0.4(..... - .....)^2 \\ &\quad +0.2(...... - ....)^2 + 0.1(..... - .....)^2 = ... \end{aligned}$$
- Standard deviation $\sigma_x = \sqrt{VAR(X)}$ = ...

- Property 1: $VAR(X+a) = VAR(X)$
  - Proof : $VAR(X+a) = E((X + ...) - E(X+a))^2$
  - $E(X+a) = ...$
  - $VAR(X+a) = E(X + ... - (......... + ...))^2 =$
- Property 2: $VAR(aX) = a^2 VAR(X)$
  - Proof : $VAR(aX) = E((....X) - E(.....X))^2$
  - $E(aX) = ...$
  - $VAR(aX) = E(aX - (...................))^2 =$

- a portfolio which a of wealth invested in asset X and $b = 1 - a$ in asset Y
- X (Y) is the rate of return of asset X (Y)
- portfolio returns: $R_p =$ ....
- example: wealth = 200, $X = 10\%$, $Y = 20\%$; a=1, a= 0.5,
- R_p is a random variable: $E(R_p) = E[aX + bY] =$ ....+ ....
- $E(VAR_p) \;=\; E[(R_p - E(R_p)]^2$

- $VAR_p = a^2 VAR(X) + b^2 VAR(Y) + 2ab[(X - E(X))(Y - E(Y)]$
- $COV(X, Y) = E[(X - E(X))(Y - E(Y))]$
- $VAR_p = a^2 VAR(X) + b^2 VAR(Y) +$ ....
- Covariance is a measure of the way in which two variables move in relation to each other
- If the covariance is positive(negative), the variables move in the same(opposite) direction
- Covariance $\rightarrow$ how a single asset contributes to portfolio risk
- COV(X,X) = ?
- Notice that the covariance with itself is variance.

- Example 2

p_i=probabilty	X(%)	Y(%)
0.2	11	-3
0.2	9	15
0.2	25	2
0.2	7	20
0.2	-2	6
1		

- $E(X) = \ldots\ldots \times 11 + \ldots\ldots \times 9 + 0.2 \times 25 + 0.2 \times 7 + 0.2 \times (-2) = 10\%$
- $E(Y) = \ldots\ldots \times (-3) + \ldots\ldots \times 15 + 0.2 \times 2 + 0.2 \times 20 + 0.2 \times (6) = 8\%$

- $E(X) = 10\%$, $E(X) = 8\%$
- $VAR(X) = \sum_{i=1}^{n} p_i(X_i - E(X))^2$
- Decimal number
- $VAR(X) = ... \times (0.11 - ....)^2 + ... \times (0.09 - ....)^2 + 0.2 \times (0.25 - 0.10)^2 + 0.2 \times (7 - 0.10)^2 + 0.2 \times ((-0.02) - 0.10)^2 = 0.0076 (\sigma_X = \sqrt{VAR(X)} = 0.0872)$
- Percentage number; $VAR(X) = 0.2(11 - 10)^2 + ... = 76$, $\sigma_X = 8.72\%$
- Decimal number
- $VAR(Y) = ... \times ((-0.03) - ....)^2 + ... \times (0.15 - ....)^2 + 0.2 \times (0.02 - 0.08)^2 + 0.2 \times (0.20 - 0.08)^2 + 0.2 \times (0.06 - 0.08)^2 = 0.00708 (\sigma_Y = \sqrt{VAR(Y)} = 0.0841)$
- Percentage number; $VAR(Y) = 70.8$, $\sigma_Y = 8.41\%$

- $COV(X,Y) = \sum_{i=1}^{n} p_i (X_i - E(X))(Y_i - E(Y))$
- Decimal Number
- $COV(X,Y) = ... \times (0.11 - ....)((-0.03) - ....) + ... \times (0.09 - ....)(0.15 - ....) + 0.2 \times (0.25 - 0.10)(0.02 - 0.08) + 0.2 \times (0.07 - 0.10)(0.20 - 0.08) + 0.2 \times ((-0.02) - 0.10)(0.06 - 0.08) = -0.0024.$
- Percentage number;
  $COV(X,Y) = 0.2(11 - 8)(-0.3 - 8) + ... = -24$

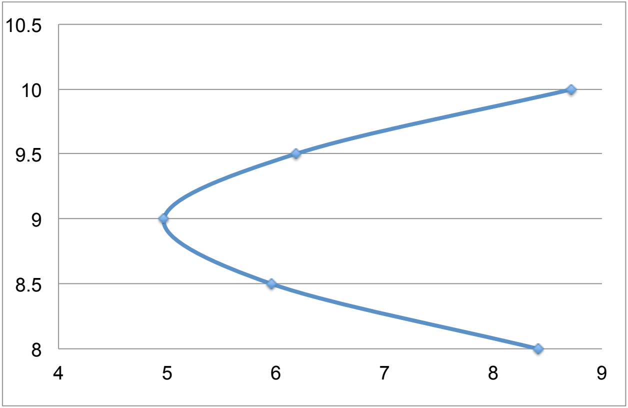
- Negative covariance implies that when X produces a return above its average return, Y do the opposite thing
- When we are losing with asset X, we are making profit with asset Y.
- Therefore, the risk is reduced if we invest in both X and Y.
- This is called “the effect of diversification” ; reducing risk by investing in a variety of assets

- Example 3 : $a = 0.50$,$b = 1 - a =$ .... , $E(X) = 0.10$, $E(Y) = 0.08$, $VAR(X) = 0.0076$ ($\sigma_X = 8.72\%$), $VAR(Y) = 0.00708(\sigma_Y = 8.41\%)$, $COV(X, Y) = -0.0024$
    - Formula : $E(R_p) =$ .... + ... and $VAR_p = a^2 VAR(X) + b^2 VAR(Y) +$ ....
    - $E(R_p) =$ .... + ... $= 9\%$
    - $VAR_p =$ .... + ... + ... $= 0.00247$ ($\sigma = \sqrt{VAR_p} = 0.0497 = 4.97\%$
- The expected return is half way between that offerred by X and by Y
- Risk is considerably less than half way between $VAR(X)$ and $VAR(Y)$

Example 4 :

Percentage in X	Percentage in Y	$E(R_p)(\%)$	$\sigma(R_p)(\%)$
100	0	10	8.72
75	25	9.5	6.18
50	50	9.0	4.97
25	75	8.5	5.96
0	100	8.0	8.41

- The relation between the portfolio return and the portfolio standard deviation



- Theoretically, the possible range of covariance extends all the way from minus infinity to plus infinity.
- We can bound it by dividing it by the product of standard deviation for the two assets
- $r_{x,y} = \frac{COV(X, Y)}{\sigma_x \sigma_y}$; $-1 \leq r_{x,y} \leq 1$
- $COV(X, Y) = r_{x,y}\sigma_x\sigma_y$
- from the example; $r_{x,y} = \frac{-0.024}{(0.0872)(0.0841)} = -0.33$
- $r_{x,y} = 1$; X, Y are perfectly correlated
- If $r_{x,y} =$ -1 ; X, Y are perfectly inversely correlated

- Example ; $r_{x,y} = \dfrac{-0.024}{(0.0872)(0.0841)} = -0.33$
- When $-1 \leq r_{x,y} < 1$ (X and Y are less than perfectly correlated), investing in both X and Y (a portfolio comprising both X and Y) is better than investing in only the single asset X or investing in only the single asset Y. The portfolio standard deviation is less than the weighted average of the component standard deviations.
- Why? benefit of diversification.

- Consider two cases : (1) perfectly correlated $r_{xy} = 1$ (2) perfectly inversely correlated $r_{xy} = -1$
- $Y = a + bX \Rightarrow b$ is positive, X and Y are perfectly correlated, $r_{xy} = 1$
- Proof : $Y = a + bX$
  $E(Y) =$ ....
  $VAR(Y) =$ ....
  $\sigma_y =$ ....
  $r_{xy} = \dfrac{COV(X,Y)}{\sigma_x \sigma_y} = \dfrac{E(X - E(X))(Y - E(Y))}{\sigma_x \sigma_y}$
  $r_{xy} \quad =$

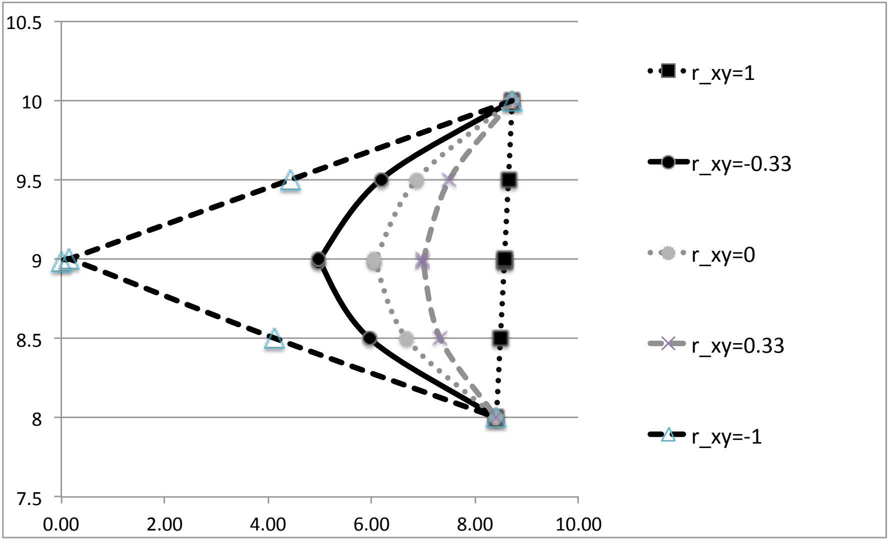
- $Y = a - bX \Rightarrow b$ is positive, X and Y are inversely perfectly correlated, $r_{xy} = -1$
- Proof : $Y = a - bX$
  $E(Y) =$ ....
  $VAR(Y) =$ ....
  $\sigma_y =$ ....
  $$r_{xy} = \frac{COV(X, Y)}{\sigma_x \sigma_y} = \frac{E(X - E(X))(Y - E(Y))}{\sigma_x \sigma_y}$$
  $$r_{xy} =$$
  $$=$$

Summary : Formula

- $E(X) = \sum_{i=1}^{n} p_i X_i$
- $VAR(X) = \sum_{i=1}^{n} p_i (X_i - E(X))^2$
- $E(R_p) = aE(X) + (1-a)E(Y)$
- $VAR(R_p) = a^2 VAR(X) + (1-a)^2 VAR(Y) + 2a(1-a)COV(X,Y)$
- $VAR(X) = \sigma_x^2$
- $VAR(R_p) = a^2\sigma_x^2 + (1-a)^2\sigma_y^2 + 2a(1-a)r_{x,y}\sigma_x\sigma_y$
- $\sigma_p = \sqrt{VAR(R_p)}$

- Relationship between expected return and standard deviation of a portfolio comprising security X and security Y
- Given $E(X) = 10\%$, $E(Y) = 8\%$, $VAR(X) = 0.0076$, $\sigma_X = 0.08717 = 8.72\%$, $VAR(Y) = 0.00708$, $\sigma_Y = 0.0841 = 8.41\%$
- Vary the value of $r_{x,y}$, find ER and VAR
- $E(R_p) = aE(X) + (1-a)E(Y)$
- $VAR(R_p) = a^2\sigma_x^2 + (1-a)^2\sigma_y^2 + 2a(1-a)r_{x,y}\sigma_x\sigma_y$
- Table 1

weight of X	$E(R_p)$	σ_p			
		$r_{x,y} = 1$	$r_{x,y} = -0.33$	$r_{x,y} = 0$	$r_{x,y} = -1$
1			8.72%	8.72%	8.72%
0.75	9.5%	8.64%	6.17%	6.87%	4.44%
0.5	9.0%	8.57%	4.96%	6.06%	0.16%
0.25	8.5%	8.49%	5.95%	6.67%	4.13%
0			8.41%	8.41%	8.41%



10.5
10
9.5
9
8.5
8
7.5
0.00
2.00
4.00
6.00
8.00
10.00
r_xy=1
r_xy=-0.33
r_xy=0
r_xy=0.33
r_xy=-1

- It can be proved that the relationship between R_p and σ_p is a straight line when $r_{x,y} = 1$
- If $r_{x,y} = -1$, the minimum variance portfolio will have zero standard deviation. (can be mathematically proved)
- $E(R_p) = aE(X) + (1 - a)E(Y)$
- $VAR(R_p) = a^2\sigma_x^2 + (1 - a)^2\sigma_y^2 + 2a(1 - a)r_{x,y}\sigma_x\sigma_y$
- (See the book, page 116 - 120 for the mathematical proof)
- If $-1 < r_{xy} < 1$, the relationship between R_p and σ_p will lie inside the triangle

- Expected return on any portfolio is the weighted avarage of the asset expected returns.
- This is not the case for standard deviation (risk).
- From the table 1

1. $r_{x,y} = 1$. Assets are perfectly positively correlated
    - There is no benefit from diversification.
    - The portfolio standard deviation is the weighted average of the component asset standard deviation.
2. $r_{x,y} = 0$. Assets are uncorrelated.
    - There is a benefit from diversification.
    - The portfolio standard deviation is less than the weighted average of the component asset standard deviation.
3. $r_{x,y} = -1$. Assets are perfectly negatively correlated.
    - Maximum advantage from diversification.
4. $-1 \leq r_{x,y} < 1$ (X and Y are less than perfectly correlated).
    - There is a benefit from diversification.
    - The smaller the correlation, the greater the risk reduction potential.

Note : Mathematical Proof

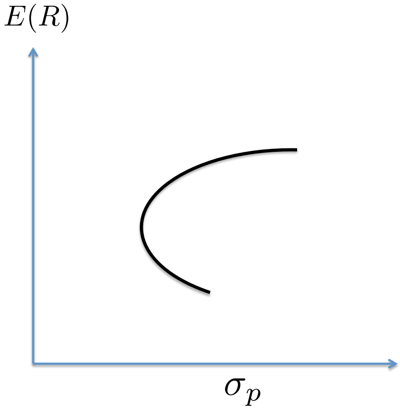
- Proof : perfectly correlated asset

$$
\begin{aligned}
E(R_p) &= aE(X) + (1-a)E(Y) \\
VAR(R_p) &= a^2\sigma_x^2 + (1-a)^2\sigma_y^2 + 2a(1-a)\sigma_x\sigma_y \\
\frac{dE(R_p)}{d\sigma_P} &= \frac{dE(R_p)/da}{d\sigma_p/da} \\
&= \frac{E(X)-E(Y)}{\sigma_x - \sigma_y}
\end{aligned}
$$

The slope of the relationship between R_p and σ_p is a constant, therefore the relationship is a straight line.

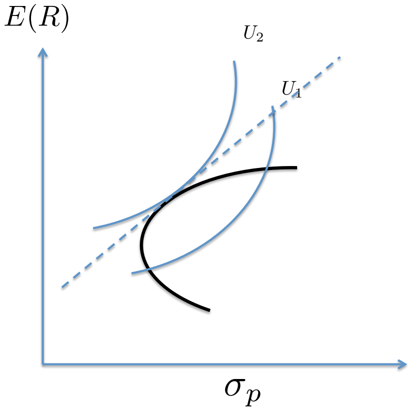
- Minimum Variance Portfolio, Given X, Y , find the value of (a, 1-a) which makes the variance of the portfolio equal to its minimum

$$
\begin{aligned}
VAR(R_p) &= a^2\sigma_x^2 + (1-a)^2\sigma_y^2 + 2a(1-a)r_{x,y}\sigma_x\sigma_y \\
\frac{dVAR(R_p)}{da} &= 0 \\
a^* &= \frac{\sigma_y^2 - r_{xy}\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2 - 2r_{x,y}\sigma_x\sigma_y}
\end{aligned}
$$



- Usually assets are less than perfectly correlated ; $-1 < r_{xy} < 1$
- The general slope of the mean-variance opportunity set is convex.
- Efficient frontier: portfolio in the opportunity set which yields highest return for a given rate of risk
- "Dominate"
- part of mean-variance opportunity set which lies above the minimum variance portfolio
- firstly developed by "Harry Markowitz"

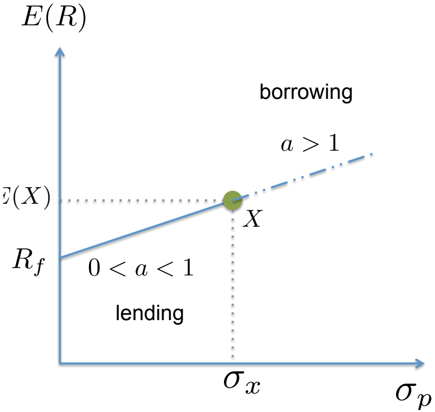
3. The Efficient Frontier with Two Risky Assets (No Risk-Free Asset)



- utility maximising portfolio is the one where indifference curve is tangent to the mean-variance opportunity set
- different investors may hold different portfolios
- (later when riskless asset is introduced, investors will hold identical combinations of risky assets)

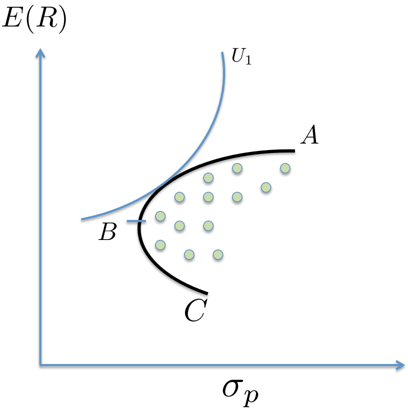
4. The Efficient Frontier with One Risky Asset and One Risk-Free Asset

- Risk-free asset, R_f, has zero variance, then the mean and the variance of the portfolio become
  - $E(R_p) = aE(X) + (1 - a)R_f$
  - $VAR(R_p) = a^2 VAR(X)$
- The relationship between expected return and standard deviation is linear



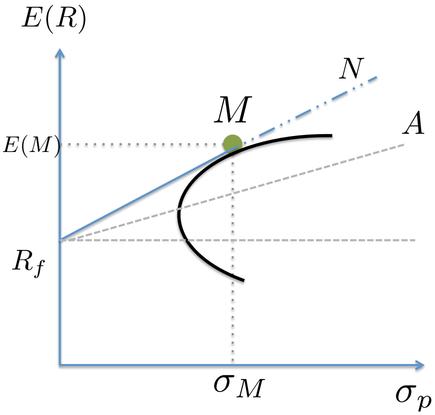
- $E(R_p) = aE(X) + (1 - a)R_f$
- $VAR(R_p) = a^2 VAR(X)$: $\sigma_p = a\sigma_X$
- The relationship between expected return and standard deviation is linear (can be mathematically proved)

5. Optimal Portfolio Choice: N risky Assets

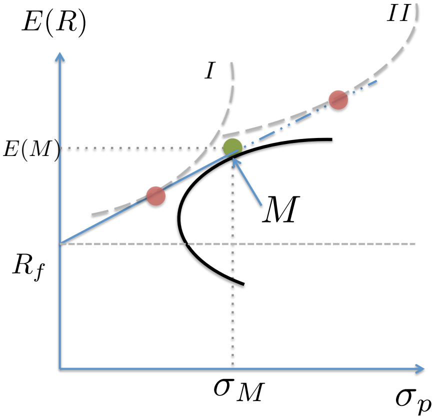


- The opportunity set with *N* risky assets has the same shape as it did with two assets.
- The only difference is that with many assets, some portfolios will fall in the interior of the opportunity set.

6. Optimal Portfolio Choice: N Assets and One Risk-Free Asset

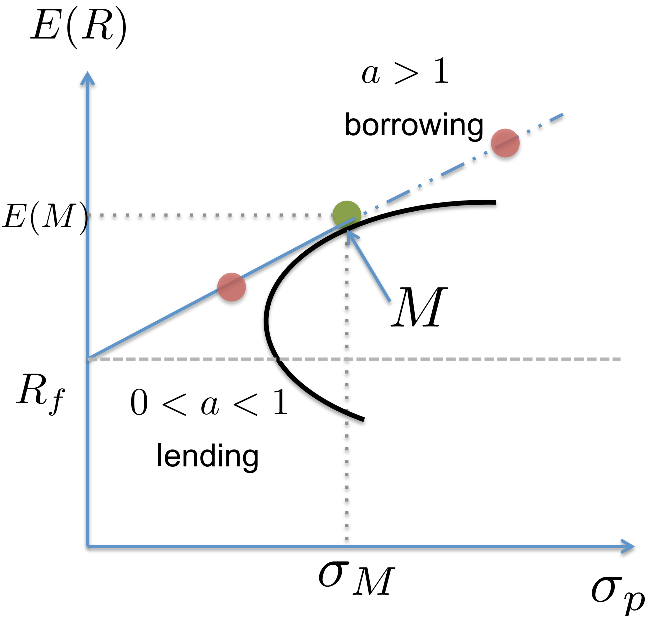


- Risk-free asset, R_f, has zero variance.
- People can borrow and lend at risk-free rate, R_f
- Portfolios along the line drawn from the risk-free rate to any feasible portfolio are possible
- Only one line dominates : the line which is tangent to the efficient frontier



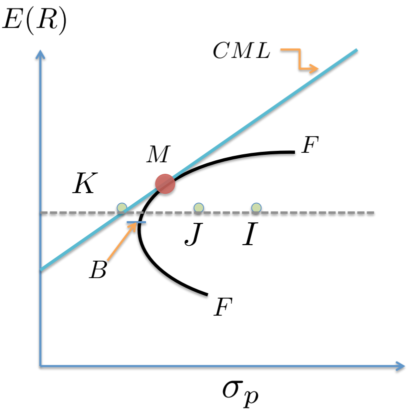
- All investors (regardless of their degree of risk aversion) will prefer combinations of the risk-free asset and portfolio M
- No investor will choose to invest in any other risky portfolio except portfolio M

- (James Tobin) All investors will hold the same asset portfolio. There are two steps in the investment selection process.
  1. choose the risky portfolio, determined by the line drawn from the risk-free rate of return tangent to the efficient frontier
  2. blend the risky portfolio by borrowing and lending depending on whether we want more risk or less risk
- Graphically illustration is provided next page


$E(R)$
$a > 1$
borrowing
$E(M)$
M
R_f
$0 < a < 1$
lending
σ_M
σ_p

- If investors have homogenous believe, they all have the same linear efficient set; Capital market line (CML)
- The equation for Capital Market Line is

7. Portfolio Diversification and Individual Asset Risk



- Should "variance" be a good measurement of "risk" ?
- Asset *I*, *J* are .... the capital market line
- People do not hold asset *I*, *J* separately
- $E(R), \sigma^2$ of asset *I*, *J* $\nrightarrow$ rate of returns the market will require from asset *I*, *J*
- Riskiness of asset *I*, *J* $\rightarrow$ their constribution in the asset portfolio (covariance risk)
- Equilibrium: CAPM , APT