Solution: Quiz 4

1. Let $P(n)$ be the predicate defined as

$$P(n) \ : \quad \sum_{i=1}^{n}(3i-1) \ = \ \frac{3n^2+n}{2}.$$

(a) Show that $P(3)$ is true.

(b) Prove by mathematical induction that $P(n)$ is true for any positive integer n.

Solution:

(a) For $n = 3$,

$$\begin{aligned} (i) \quad \sum_{i=1}^{n}(3i-1) \ &= \ \sum_{i=1}^{3}(3i-1) \\ &= \ [3(1)-1]+[3(2)-1]+[3(3)-1] = 2+5+8 = 15, \end{aligned}$$

and
(ii) $(3n^2+n)/2 = (3(3^2)+3)/2 = 15$.
From (i) and (ii), $\sum_{i=1}^{4}(4i-1) = 2(4^2)+4$, which implies that P(4) is true. ■

(b) We want to show that $P(n)$ is true for $n \geq 1$.
(I) **Basis step:** $P(1)$: $\sum_{i=1}^{1}(3i-1) \ = \ (3 \cdot 1^2+1)/2$ is true because $\sum_{i=1}^{1}(3i-1) = (3(1)-1) = 2$ and $(3 \cdot 1^2+1)/2 = 4/2 = 2$.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$.
Assume that $P(k)$ is true: $\sum_{i=1}^{k}(3i-1) \ = \ (3k^2+k)/2$. ———($\bigstar$) "inductive hypothesis"
We want to show that $P(k+1)$: "$\sum_{i=1}^{k+1}(3i-1) \ = \ [3(k+1)^2+(k+1)]/2$. "

Note that $3(k+1)^2+(k+1) = (3(k+1)+1)(k+1) = (3k+4)(k+1) = 3k^2+7k+4$. So, we can show that $P(k+1)$ is true by showing that:

$$P(k+1) \ : \ \sum_{i=1}^{k+1}(4i-1) = \frac{3k^2+7k+4}{2}. \tag{1}$$

Consider the left-hand side of $P(k+1)$:

$$\begin{aligned} \sum_{i=1}^{k+1}(3i-1) \ &= \ \underbrace{\sum_{i=1}^{k}(3i-1)}_{(\bigstar)} + [3(k+1)-1] \ = \ \frac{3k^2+k}{2} + [3k+2] \text{ by } (\bigstar) \\ &= \ \frac{[3k^2+k]+2[3k+2]}{2} = \frac{3k^2+7k+4}{2}, \end{aligned}$$

which is the same as 1. That is, $P(k+1)$ is true.
Hence, from (I) basis step and (II) inductive step, $P(n)$ is proved for any integer $n \geq 1$ by mathematical induction. ■