Solution: Quiz 6

1. Consider the functions $f : D_f \to \mathbb{R}$ and $g : X \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 \quad , & x < 0 \\ x-1, & 0 \le x < 8. \end{cases} \qquad \text{and} \qquad g(x) = \sqrt{x} + 5$$

where $D_f \subset \mathbb{R}$, $X = \{0, 1, 4, 9, 16, 25\}$, and $\mathbb{R}$ is the set of real numbers.

Determine whether or not $f \circ g$ can be defined (by explaining your answer in terms of domain and range). If so, find $f \circ g$ and its domain.

Solution: Let D_f and D_g be the domains of f and g, respectively and let R_f and R_g be the ranges of f and g, respectively.

From the definition of $f(x)$, the domain of f is $\boxed{D_f = (-\infty, 8)}$.
From the definition of $g(x)$, $x \in X = \{0, 1, 4, 9, 16, 25\}$ and the domain of g is $\boxed{D_g = X}$.

To find R_g, for $x \in D_g = \{0, 1, 4, 9, 16, 25\}$ and $g(x) = \sqrt{x} + 5$, since $g(0) = \sqrt{0} + 5 = 5$, $g(1) = \sqrt{1} + 5 = 6$, $g(4) = \sqrt{4} + 5 = 7$, $g(9) = \sqrt{9} + 5 = 8$, $g(16) = \sqrt{16} + 5 = 9$, $g(25) = \sqrt{25} + 5 = 10$, then $\boxed{R_g = \{5, 6, 7, 8, 9, 10\}}$.

Therefore, $\boxed{f \circ g}$ can be defined because the following condition holds:

$$R_g \cap D_f = \{5, 6, 7, 8, 9, 10\} \cap (-\infty, 8) = \{5, 6, 7\} \neq \emptyset.$$

Notice that R_g is not a subset of D_f and we may **not** have $D_{f \circ g} = D_g$ and we only know that $D_{f \circ g} \subseteq D_g$.
The followings are two possible approaches for finding $f \circ g$ and its domain.

Approach I

To find $f \circ g$,

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x} + 5),$$

we must use x such that $x \in D_g = \{0, 1, 4, 9, 16, 25\}$ and $\sqrt{x} + 5 \in R_f = (-\infty, 8)$, i.e.

$$\sqrt{x} + 5 < 8 \quad \Rightarrow \quad \sqrt{x} < 8 - 5 \quad \Rightarrow \quad \sqrt{x} < 3 \quad \Rightarrow \quad 0 \le x < 9 \quad \Rightarrow \quad x \in [0, 9).$$

That is, the domain $D_{f \circ g}$ is the set of x such that

$$x \in \{0, 1, 4, 9, 16, 25\} \cap [0, 9) = \{0, 1, 4\} \qquad \Rightarrow \qquad D_{f \circ g} = \{0, 1, 4\}.$$

Therefore, the composite function $f \circ g$ is

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x} + 5) = [\sqrt{x} + 5] - 1 = \sqrt{x} + 4$$

for $x \in D_{f \circ g} = \{0, 1, 4\}$. ■

Approach II

Consider the composite function $f \circ g$:

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x} + 5).$$

Note that we can only define the composite function when x satisfies $\sqrt{x} + 5 \in \boxed{D_f = (-\infty, 8)}$ from the definition of f.

For $x \in D_g = X = \{0, 1, 4, 9, 16, 25\}$,

$x = 0$: $(f \circ g)(0) = f(g(0)) = f(\sqrt{0} + 5) = f(5) = 5 - 1 = 4$, since $5 \in [0, 8)$,

$x = 1$: $(f \circ g)(1) = f(g(1)) = f(\sqrt{1} + 5) = f(6) = 6 - 1 = 5$, since $6 \in [0, 8)$,

$x = 4$: $(f \circ g)(4) = f(g(4)) = f(\sqrt{4} + 5) = f(7) = 7 - 1 = 6$, since $7 \in [0, 8)$,

$x = 9$: $(f \circ g)(9) = f(g(9)) = f(\sqrt{9} + 5) = f(8)$ is undefined , since $8 \notin D_f = (-\infty, 8)$,

$x = 16$: $(f \circ g)(16) = f(g(16)) = f(\sqrt{16} + 5) = f(9)$ is undefined, since $9 \notin D_f = (-\infty, 8)$,

$x = 25$: $(f \circ g)(25) = f(g(25)) = f(\sqrt{25} + 5) = f(10)$ is undefined, since $10 \notin D_f = (-\infty, 8)$,

That is, the only values of x that is possible to use for $f \circ g$ are $x = 0, 1, 4$ and, therefore, the domain $D_{f \circ g} = \{0, 1, 4\}$. Alternatively, $f \circ g$ can be represented by the following arrow diagram and its domain $D_{f \circ g} = \{0, 1, 4\}$ can be seen from the values x that can be mapped through g and f.