

Optimization with Equality Constraints II

Pawin Siriprapanukul

CW Ch.12

Outline

- Utility Maximization and Consumer Demand
- Homogeneous Functions*
- Least-Cost Combination of Inputs*

Utility Maximization and Consumer Demand

- A consumer chooses to consume between two goods, both of which have continuous, positive marginal-utility functions.
- The prices of both goods are market-determined, hence exogenous.
- Suppose the total budget of the consumer is B , the problem of the consumer becomes

$$\begin{aligned} \max_{x,y} U &= U(x, y) \\ \text{s. t. } xP_x + yP_y &= B, \end{aligned}$$

where $U_x, U_y > 0$.

Utility Maximization and Consumer Demand: First-Order Condition

- The Lagrangian equation of this optimization problem is

$$L = U(x, y) + \lambda(B - xP_x - yP_y).$$

- FOC:

$$L_x = U_x - \lambda P_x = 0$$

$$L_y = U_y - \lambda P_y = 0$$

$$L_\lambda = B - xP_x - yP_y = 0.$$

- From the first two equations, we get

$$\frac{U_x}{P_x} = \frac{U_y}{P_y} = \lambda.$$

Utility Maximization and Consumer Demand: First-Order Condition

- In order to maximize utility, consumers must allocate their budgets so as to equalize the ratio of marginal utility to price for every commodity.
- At the optimum, these ratios should have the common value λ^* .
- In the present context, $\lambda^* = \partial U^* / \partial B$ that is the optimal value of the Lagrange multiplier can be interpreted as the **marginal utility of money** (budget money) when the consumer's utility is maximized.

Example

- Previously, the solution of the problem

$$\max_{x_1, x_2} U = x_1 x_2 + 2x_1$$

$$\text{s. t.} \quad 4x_1 + 2x_2 = 60$$

is $(x_1^*, x_2^*, \lambda^*) = (8, 14, 4)$ and $U^* = L^* = 128$.

- At this optimum point,

$$U_1 =$$

$$U_2 =$$

$$\frac{U_1}{P_1} =$$

$$\frac{U_2}{P_2} =$$

Example (Continued)

- Solve the problem

$$\max_{x_1, x_2} U = x_1 x_2 + 2x_1$$

$$\text{s. t.} \quad 4x_1 + 2x_2 = 61,$$

and calculate for $U^*(= L^*)$.

Utility Maximization and Consumer Demand: First-Order Condition

- We can restate the FOC of utility maximization problem the in the form

$$\frac{U_x}{U_y} = \frac{P_x}{P_y},$$

which can be related to the indifference curve.

- Recall that an *indifference curve* is defined as the locus of combinations (x, y) that yield a constant level of U . This means that

$$dU = U_x dx + U_y dy = 0$$

with implies that $dy/dx = -U_x/U_y$.

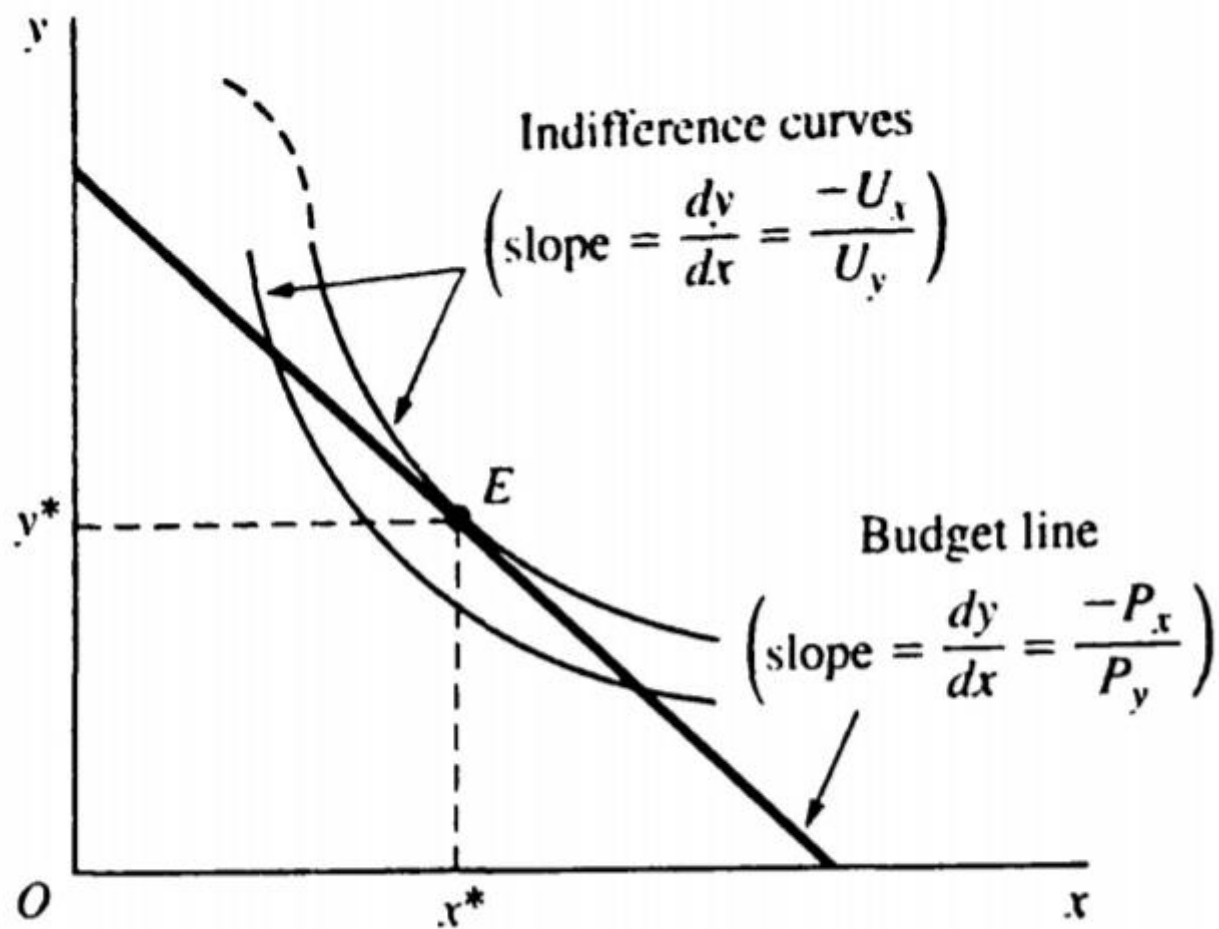
- From the budget constraint $xP_x + yP_y = B$, we can rewrite the function as

$$y = \frac{B}{P_y} - \frac{P_x}{P_y}x$$

so that the slope is $-P_x/P_y$.

Utility Maximization and Consumer Demand: First-Order Condition

- In order to maximize utility, a consumer must allocate the budget such that the slope of the budget line (the rate of exchange in the market) is equal to the slope of an indifference curve (the rate of exchange from his preferences).
- This condition is met at the point where the budget line is tangent to an indifference curve.



(a)

Utility Maximization and Consumer Demand: Second-Order Condition

- It can be shown that the second derivative of an indifference curve is

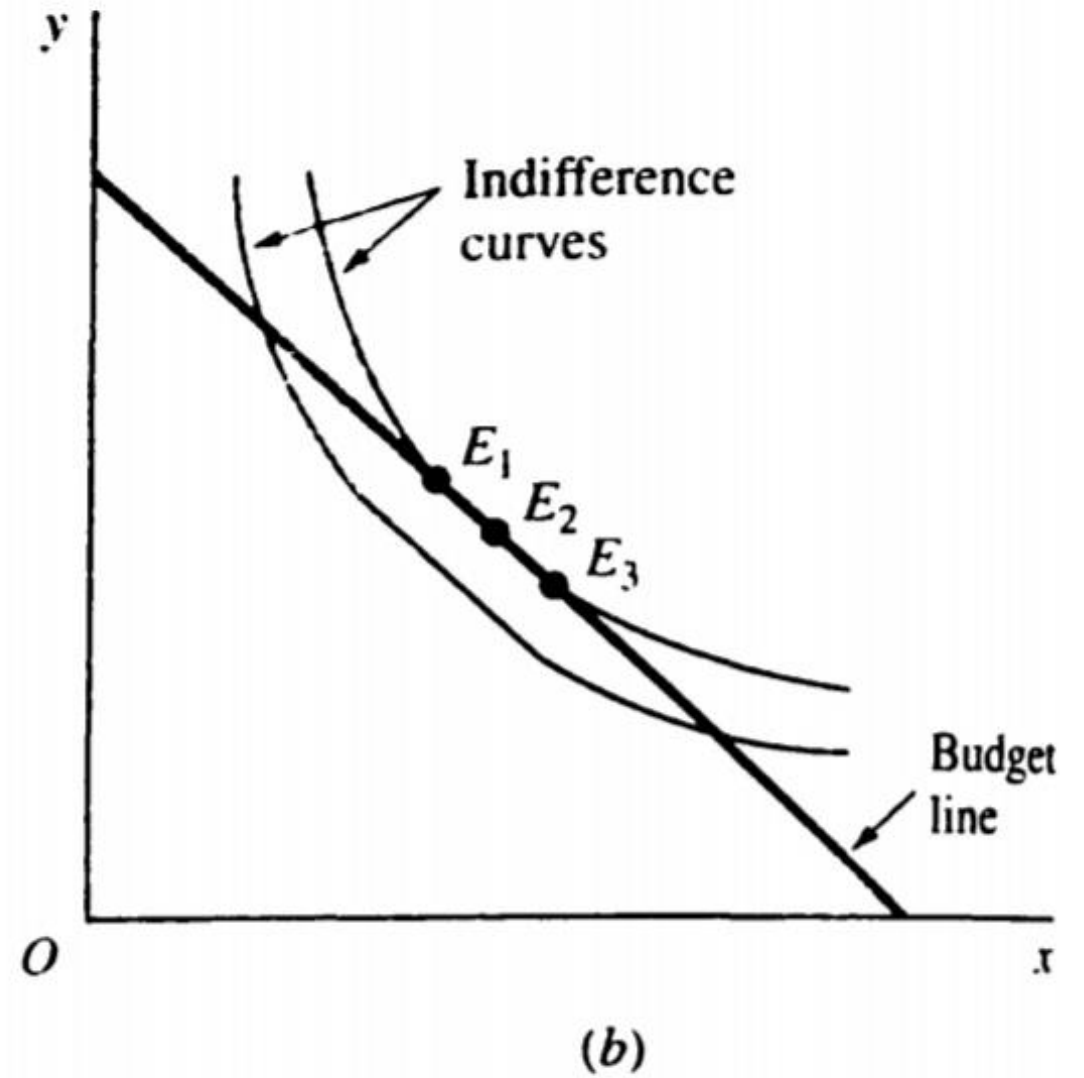
$$\frac{d^2y}{dx^2} = \frac{|\bar{H}|}{U_y P_y^2}.$$

Since U_y must be positive, the sign of $\frac{d^2y}{dx^2}$ depends on the sign of $|\bar{H}|$.

- For this utility maximization problem, the bordered Hessian $|\bar{H}|$ must be positive. The positive bordered Hessian implies that the second derivative $\frac{d^2y}{dx^2}$ is also positive, which makes the indifference curve to be a ***strictly convex*** curve.

Utility Maximization and Consumer Demand: Second-Order Condition

- Recall that our second-order condition is just *sufficient*. Even when it is not satisfied, we may face with the maximization case.
- Consider diagram on the right, here at points E_1 , E_2 and E_3 we have $|H| = 0$. However, these can be the maximum utility that the consumer can attain given budget constraint.



Utility Maximization and Consumer Demand: Comparative-Static Analysis

- In our consumer model, the prices P_x and P_y are exogenous, as is the amount of total budget B .
- If we assume the satisfaction of the second-order condition, we can analyze the comparative-static properties of the model on the basis of the first-order condition, viewed as a set of equations $F^j = 0, j = 1, 2, 3$.
- Assume that the implicit function theorem is applicable, we may express the optimal values of the endogenous variables as implicit functions of the exogenous variables

$$x^* = x^*(P_x, P_y, B)$$

$$y^* = y^*(P_x, P_y, B)$$

$$\lambda^* = \lambda^*(P_x, P_y, B).$$

These are known to process continuous derivatives that give comparative-static information.

Utility Maximization and Consumer Demand: Comparative-Static Analysis

- From the first-order condition, we substitute the optimal solutions to get

$$B - x^*P_x - y^*P_y = 0$$

$$U_x(x^*, y^*) - \lambda^*P_x = 0$$

$$U_y(x^*, y^*) - \lambda^*P_y = 0.$$

- Suppose we want to study the change in optimal solutions caused by the change in total budget B

$$-P_x \frac{\partial x^*}{\partial B} - P_y \frac{\partial y^*}{\partial B} = -1$$

$$-P_x \frac{\partial \lambda^*}{\partial B} + U_{xx} \frac{\partial x^*}{\partial B} + U_{xy} \frac{\partial y^*}{\partial B} = 0$$

$$-P_y \frac{\partial \lambda^*}{\partial B} + U_{yx} \frac{\partial x^*}{\partial B} + U_{yy} \frac{\partial y^*}{\partial B} = 0.$$

Utility Maximization and Consumer Demand: Comparative-Static Analysis

- Rearrange the previous system of equations into matrix form, we get

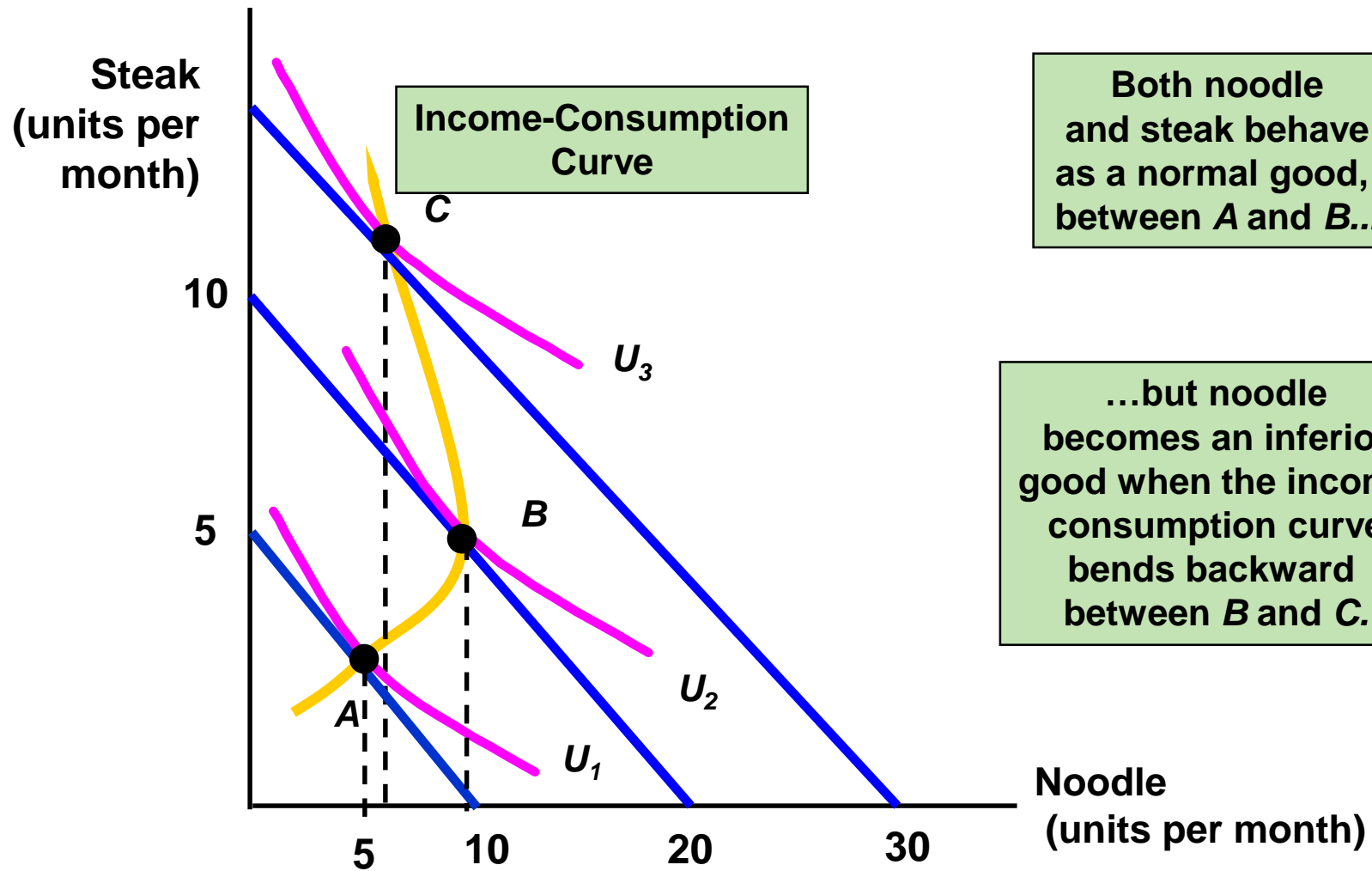
$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{bmatrix} \begin{bmatrix} \partial \lambda^* / \partial B \\ \partial x^* / \partial B \\ \partial y^* / \partial B \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

- Applying Cramer's rule, we can solve for the following two derivatives

$$\frac{\partial x^*}{\partial B} = \frac{1}{|\bar{H}|} \begin{vmatrix} 0 & -1 & -P_y \\ -P_x & 0 & U_{xy} \\ -P_y & 0 & U_{yy} \end{vmatrix} = \frac{1}{|\bar{H}|} \begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix}$$

$$\frac{\partial y^*}{\partial B} = \frac{1}{|\bar{H}|} \begin{vmatrix} 0 & -P_x & -1 \\ -P_x & U_{xx} & 0 \\ -P_y & U_{yx} & 0 \end{vmatrix} = \frac{-1}{|\bar{H}|} \begin{vmatrix} -P_x & U_{xx} \\ -P_y & U_{yx} \end{vmatrix}.$$

- These tell whether the products are **normal goods** or **inferior goods**.



Utility Maximization and Consumer Demand: Comparative-Static Analysis

- Now, suppose we want to analyze the effect of a change in P_x , we can obtain the following system

$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{bmatrix} \begin{bmatrix} \partial \lambda^* / \partial P_x \\ \partial x^* / \partial P_x \\ \partial y^* / \partial P_x \end{bmatrix} = \begin{bmatrix} x^* \\ \lambda^* \\ 0 \end{bmatrix}.$$

- Applying Cramer's rule, we can solve for

$$\frac{\partial x^*}{\partial P_x} = \frac{1}{|\bar{H}|} \begin{vmatrix} 0 & x^* & -P_y \\ -P_x & \lambda^* & U_{xy} \\ -P_y & 0 & U_{yy} \end{vmatrix} = -\frac{x^*}{|\bar{H}|} \begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix} + \frac{\lambda^*}{|\bar{H}|} \begin{vmatrix} 0 & -P_y \\ -P_y & U_{yy} \end{vmatrix}$$

$$\frac{\partial y^*}{\partial P_x} = \frac{1}{|\bar{H}|} \begin{vmatrix} 0 & -P_x & x^* \\ -P_x & U_{xx} & \lambda^* \\ -P_y & U_{yx} & 0 \end{vmatrix} = \frac{x^*}{|\bar{H}|} \begin{vmatrix} -P_x & U_{xx} \\ -P_y & U_{yx} \end{vmatrix} - \frac{\lambda^*}{|\bar{H}|} \begin{vmatrix} 0 & -P_x \\ -P_y & U_{yx} \end{vmatrix}.$$

Utility Maximization and Consumer Demand: Comparative-Static Analysis

- Consider the first equation

$$\frac{\partial x^*}{\partial P_x} = -\frac{x^*}{|\bar{H}|} \begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix} + \frac{\lambda^*}{|\bar{H}|} \begin{vmatrix} 0 & -P_y \\ -P_y & U_{yy} \end{vmatrix}.$$

- The first term can be seen as the **income effect** of price change as

$$-\frac{x^*}{|\bar{H}|} \begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix} = -\left(\frac{\partial x^*}{\partial B}\right) x^* = \left(\frac{\partial x^*}{\partial B}\right) \frac{dB}{dP_x}.$$

This term can be either positive or negative.

Utility Maximization and Consumer Demand: Comparative-Static Analysis

- The second term can be seen as the **substitution effect**. To see this, we go back to the derivatives of the first order condition assuming that the change of total budget caused by the change in P_x is zero, or

$$-P_x \frac{\partial x^*}{\partial P_x} - P_y \frac{\partial y^*}{\partial P_x} = 0.$$

- This changes the equation system to be

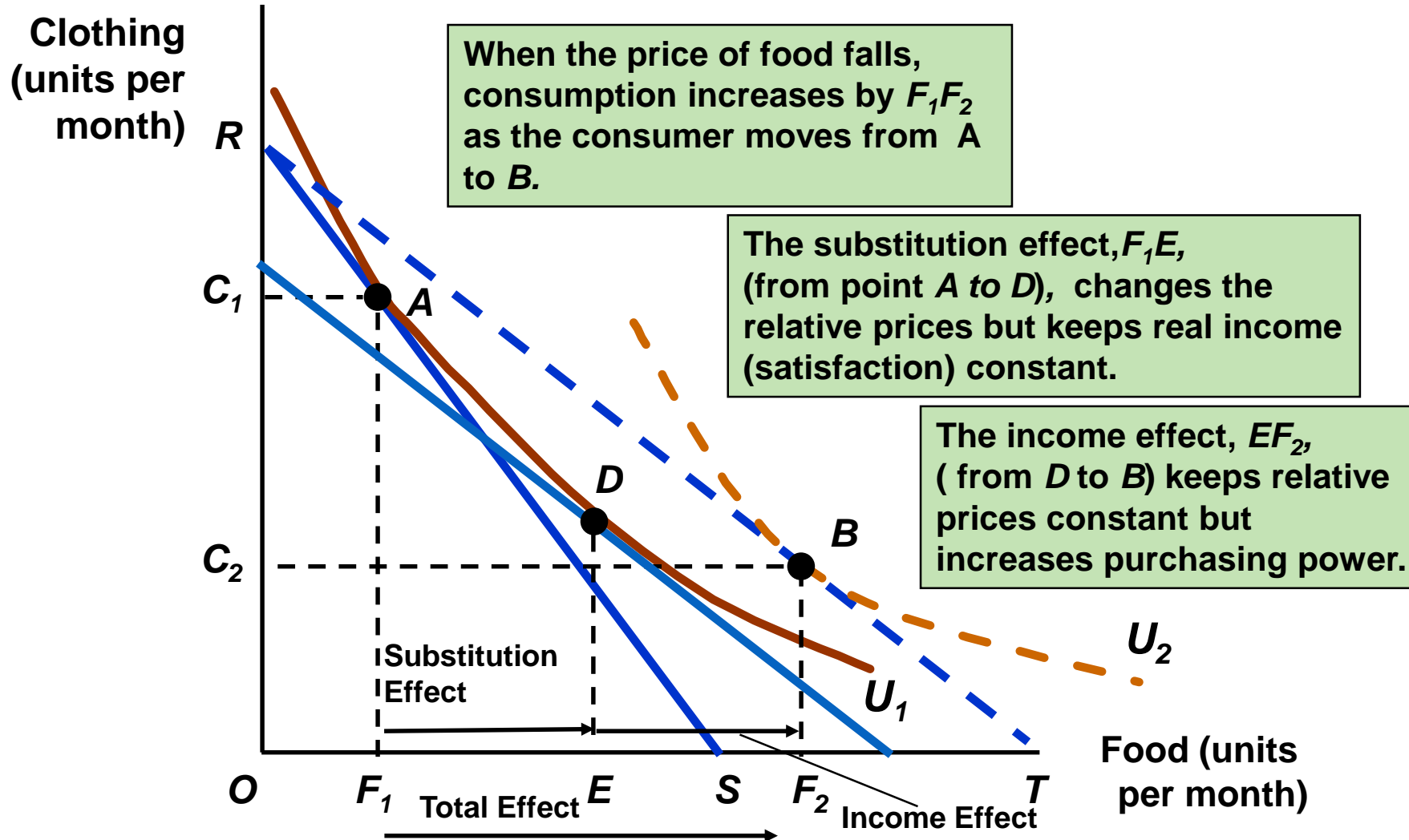
$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{bmatrix} \begin{bmatrix} \partial \lambda^* / \partial P_x \\ \partial x^* / \partial P_x \\ \partial y^* / \partial P_x \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda^* \\ 0 \end{bmatrix}.$$

- From this, we have

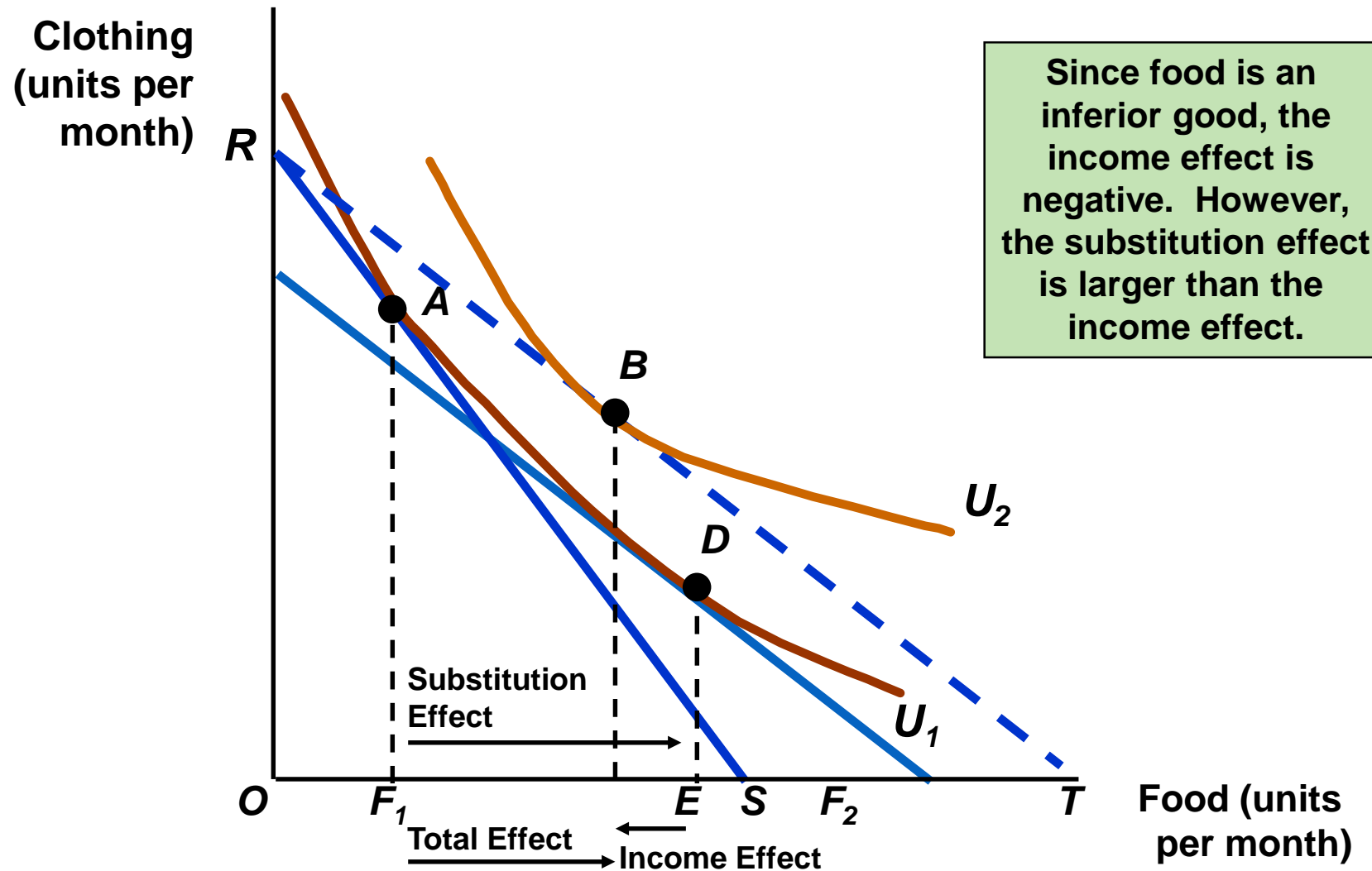
$$\left(\frac{\partial x^*}{\partial P_x} \right)_{\text{compensated}} = \frac{1}{|\bar{H}|} \begin{vmatrix} 0 & 0 & -P_y \\ -P_x & \lambda^* & U_{xy} \\ -P_y & 0 & U_{yy} \end{vmatrix} = \frac{\lambda^*}{|\bar{H}|} \begin{vmatrix} 0 & -P_y \\ -P_y & U_{yy} \end{vmatrix},$$

which is the second term that we are considering. This term is always **negative**.

Income and Substitution Effects: Normal Good



Income and Substitution Effects: Inferior Good



Utility Maximization and Consumer Demand: Comparative-Static Analysis

- From the second equation

$$\frac{\partial y^*}{\partial P_x} = \frac{x^*}{|\bar{H}|} \begin{vmatrix} -P_x & U_{xx} \\ -P_y & U_{yx} \end{vmatrix} - \frac{\lambda^*}{|\bar{H}|} \begin{vmatrix} 0 & -P_x \\ -P_y & U_{yx} \end{vmatrix},$$

this can be seen as the decomposition of income and substitution effects as well.