1. Integral Calculus: Economic Applications

(1) From marginal to Total Functions

$$\begin{tikzcd}
\boxed{\begin{matrix} TC \\ TR \end{matrix}} \arrow[r, "\text{diff}"] & \boxed{\begin{matrix} MC \\ MR \end{matrix}} \arrow[l, bend left, "\text{Integral}"]
\end{tikzcd}
\qquad \frac{dC}{dQ} \quad \frac{dR}{dQ}$$

$$\boxed{\text{Saving } f^{n}} \xrightarrow{\text{diff}} MPS = \frac{dS}{dY}$$

$$\boxed{\text{Consumption } f^{n}} \xrightarrow{\text{diff}} MPC = \frac{dC}{dY}$$

Ex Assuming the MPS

$$S'(Y) = 0.3 - 0.1Y^{-\frac{1}{2}}$$

where Income $(Y) = 81 \Rightarrow S(Y) = 0$

Find $S(Y)$

$$S(Y) = \int S'(Y)\,dY$$

$$= \int \left[0.3 - 0.1Y^{-\frac{1}{2}}\right] dY$$

$$= 0.3Y - 0.2Y^{\frac{1}{2}} + S_0$$

$$S(81) = 0.3(81) - 0.2\sqrt{81} + S_0$$

$$0 = 0.3(81) - 0.2(9) + S_0$$

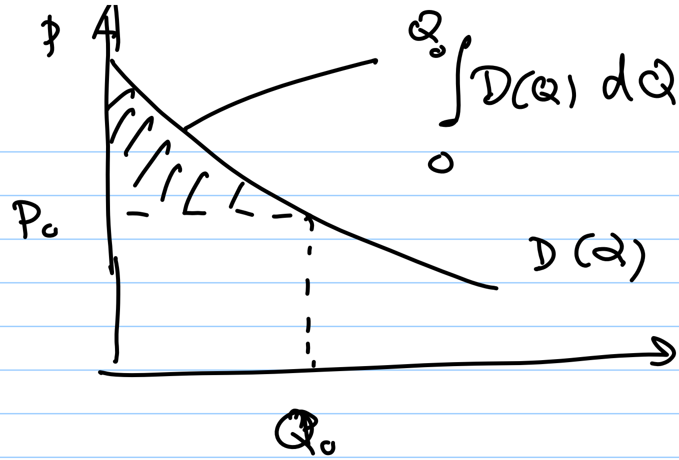
$$= 24.3 - 1.8 + S_0$$

$$0 = 22.5 + S_0$$

$$S_0 = -22.5$$

$$\therefore \; S(Y) = 0.3Y - 0.2\sqrt{Y} - 22.5$$

(2) Consumer & Producer Surpluses.



$$\int_0^{Q_0} D(Q)\, dQ - P_0 Q_0$$

(3) Investment and Capital Formation

$K(t)$ = capital stock.

$\dfrac{dK(t)}{dt}$ = the rate of capital formation @ time t

$=$ the rate of net investment flows @ time t

$= I(t)$

31 Dec 2014 — 31 Dec 2015

$K(2014)$ — $K(2015)$

$$\boxed{I(t) = \frac{dK(t)}{dt}} \rightarrow I(t) \text{ is the derivative of } K(t)$$

$$\int I(t)\,dt = \int \frac{dK(t)}{dt}\,dt$$

$$\boxed{\int I(t)\,dt = K(t)}$$ $K(t)$ is the antiderivative/ integral of $I(t)$

Note $I(t) = I_g(t) - \delta K(t)$

where $\delta =$ the depreciation rate of capital

Ex $I(t) = 4t^{\frac{3}{2}}$

The initial capital stock @ $t=0$ is $K(0)$

Find the capital stock fn

$\leftrightarrow$ the time path of capital stock

$$K(t) = \int I(t)\,dt$$

$$= \int 4t^{\frac{3}{2}}\,dt$$

$$= \frac{4t^{\frac{3}{2}+1}}{\frac{3}{2}+1} + k_0$$

$$= \frac{4t^{\frac{5}{2}}}{\frac{5}{2}} + k_0$$

$$K(t) = \frac{8}{5}t^{\frac{5}{2}} + k_0$$

@ $t=0$ $\quad K(0) = \frac{8}{5}(0)^{\frac{5}{2}} + k_0$

$$K(0) = k_0$$

$$\therefore K(t) = \frac{8}{5}t^{\frac{5}{2}} + K(0)$$

$$\int_a^b I(t)\,dt = K(t)\Big|_a^b = K(b) - K(a)$$

$$\boxed{\int_0^t I(t)\,dt = K(t) - K(0)}$$

the amount of capital accumulation during $[0, t]$

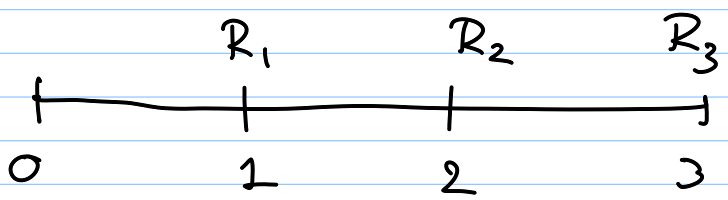
$$\boxed{K(t) = K(0) + \int_0^t I(t)\,dt}$$

Ex $\displaystyle\int_1^4 I(t)\,dt$ where $I(t) = 3t^2$

$= \underline{\underline{14}}$

(4) Present Value of a Cash Flow.

ⓐ A Discrete Case



$$A = \frac{R_1}{1+i} + \frac{R_2}{(1+i)^2} + \frac{R_3}{(1+i)^3}$$

$i =$ interest rate per annum

ⓑ A continuous Case

't' is taken as a continuous variable

$R(t)$ is continuous revenue stream

$$A_p = \int_0^3 R(t)e^{-rt}\,dt$$

accumulated PV

r = interest rate, compound continuously

In general

$$A = \int_0^T R(t)e^{-rt}\,dt$$

from now until T years

Ex: The wine dealer's problem

C = the cost of the care of wine,

incured at present time

S = the (constant) storage cost,

which is a stream cost

(dollars per year)

The total PV of the storage cost incurred

in a total of t year is

$$\int_0^t s e^{-rt}\,dt = s\int_0^t e^{-rt}\,dt$$

$$= s\left[-\frac{1}{r}e^{-rt}\right]_0^t$$

$$= s\left[\left(-\frac{1}{r}e^{-rt}\right) - \underbrace{\left(-\frac{1}{r}e^{-r0}\right)}_{+\frac{1}{r}}\right]$$

$$= \frac{s}{r}\left[1 - e^{-rt}\right]$$

$V(t)$ = the future sale (value) of this case of wine

$$\underline{N(t)} = \underbrace{V(t)e^{-rt}}_{TR} - \underbrace{\left[\frac{s}{r}(1-e^{-rt}) + C\right]}_{TC}$$

Net PV of this case of wine

Find the time to sell this case of wine to get max Net PV.

F.O.C. $N'(t) = \left[V'(t)e^{-rt} - rV(t)e^{-rt}\right] - \frac{s}{\cancel{r}}(\cancel{r})e^{-rt}$

$$0 = \underbrace{\left[V'(t) - rV(t) - s\right]}\,\underbrace{e^{-rt}}_{\neq 0}$$

If we know $V(t)$ $\Rightarrow$ we can solve for

t^*

(5) The PV of Perpetual Flow

If the cash flow persists forever, i.e., revenue from land

$$A = \int_0^\infty R(t)\, e^{-rt}\, dt$$

$R(t) = R$

$$A = \int_0^{\infty} R e^{-rt}\, dt$$

$$= \lim_{b \to \infty} \int_0^b R e^{-rt}\, dt$$

$$= \lim_{b \to \infty} \left[\frac{R}{r} e^{-rt} \right]_0^b$$

$$= \lim_{b \to \infty} \left[\frac{R}{r} (1 - e^{-rt}) \right]$$

check

$$= \frac{R}{r}$$

(6) Domar Growth Model